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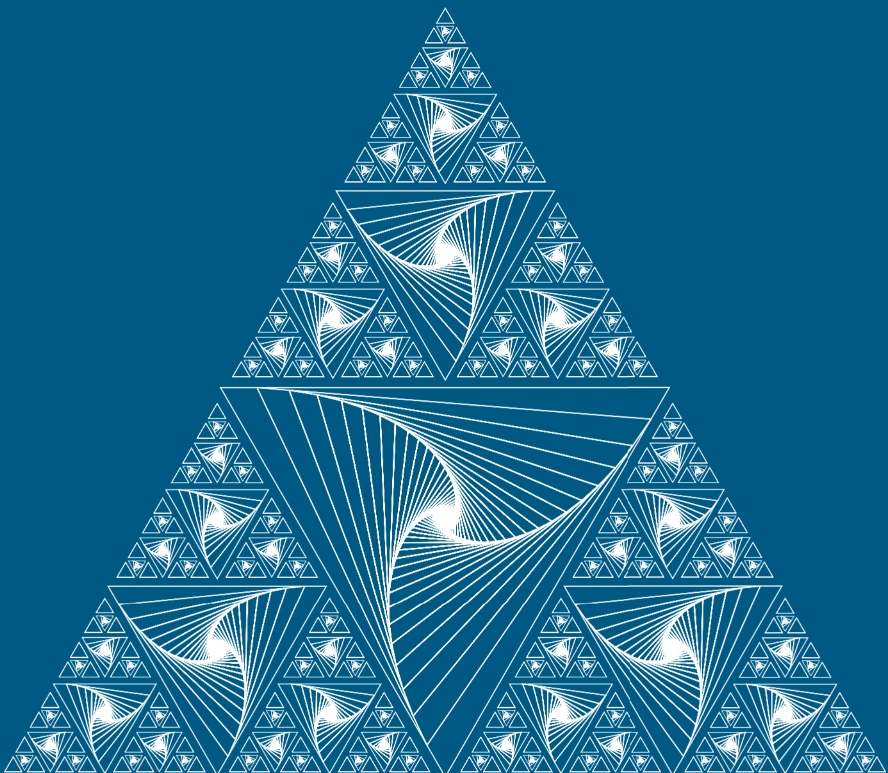
MATHEMATICS MAGAZINE



- Sequences of multi-polygonal numbers
- Changing seats on the bus: permutations, bijections, and cycles
- Coefficients of cyclotomic polynomials
- The Box Problem revisited: three-factor polynomials and a Diophantine equation

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COVER IMAGE

Binomial Pursuit © 2016 David A. Reimann (*Albion College*). Used by permission.

Influenced by several binomial items in this issue, the artwork depicts a Sierpinski triangle where a pursuit triangle is used on the interior triangles, and a simple black outlined triangle is used on the exterior triangles. The rotation of the pursuit reverses at each scaling at the lowest level.

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MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

Triangular, square, and pentagonal numbers are examples of polygonal numbers. Multi-polygonal numbers are numbers that are polygonal in more than one way, such as 36, which is both square ($36 = 6^2$) and triangular ($36 = 1 + 2 + \cdots + 8$). In Multi-Polygonal Numbers, Roger Nelsen develops methods to generate sequences of multi-polygonal numbers such as the sequence of all square triangular numbers (for which 36 is the second smallest such number).

Vladimir Pozdnyakov and Michael Steele create a stylized story about disregarding assigned seats on a bus to consider questions about bumping passengers and the associated bijections, permutations, and cycles.

At times a mathematician's intuition can be fooled. That is the case when considering the coefficients of cyclotomic polynomials, those polynomials that are factors of $x^n - 1$. Gary Brookfield introduces cyclotomic polynomials and explains that the 105th cyclotomic polynomial has the coefficient of -2 for both x^{41} and x^7, proving that all coefficients for the first 104 cyclotomic polynomials are in $\{-1, 0, 1\}$.

Motivated by a student's question, Jathan Austin determines when the sum of two integers is equal to the difference of their least common multiple and greatest common divisor.

How do you prove something when you don't know the answer? Michael Spivey uses basic properties of two-variable triangular recurrence relations to show that binomial coefficients satisfy the well-known binomial recurrence.

The symmedian point is a type of center of a triangle; it has also been called the Lemoine point and Grebe's point. Majid Bani-Yaghoub, Noah Rhee, and Jawad Sadek construct a least-squares problem for which the symmedian point is the solution.

David Cruz-Uribe and Gregory Convertito revisit and generalize a classic calculus problem about building the biggest box (and when it has rational side lengths) and the relationship to integer solutions of Diophantine equations.

Morrie's law is a curious identity involving the product of cosines. In A Geometric Proof of a Morrie-Type Formula, Samuel Moreno and Esther Garcìa-Caballero use the regular heptagon and add some of its diagonals and angle bisectors to prove a Morrie-type formula for the product of the cosines of $\frac{\pi}{7}$, $\frac{2\pi}{7}$, and $\frac{3\pi}{7}$.

Diamonds with many facets are desirable because of the light reflected off the many cuts. Shedding light is the reason why mathematicians often prove a result in a number of ways. Stephen Kaczkowski considers three proofs of $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^k = \frac{e}{e-1}$. One proof is calculus-based, the second uses real analysis, while the third applied Tannery's theorem.

In between the articles, Roger Nelsen, Stanley Huddy, Grégoire Nicollier, and Gopal Panda and Ravi Davala offer proofs without words while Tom Edgar and Angel Plaza each provide two such proofs. Amy and David Reimann interview Robert Fathauer, touching on unusual dice, the use of computers to create art, and ceramics. Brendan Sullivan has constructed a puzzle to get you in the mood for this summer's MathFest in Columbus, OH. The issue concludes with the Problems and Reviews. Highlights include reviews of the documentary *Navajo Math Circles*, of a new magazine for undergraduate math students (*Chalkdust*), and of former MAGAZINE editor Frank Farris' *Creating Symmetry: The Artful Mathematics of Wallpaper Patterns*.

Michael A. Jones, Editor

ARTICLES

Multi-Polygonal Numbers

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A polygonal number is a positive integer that can be represented by objects arranged in the shape of a polygon. Examples, which date to the time of the ancient Greek geometers, include the triangular, square, oblong, and pentagonal numbers. For $n \geqslant 1$ the triangular numbers are $T_n = 1 + 2 + \cdots + n = n(n+1)/2$ (think of a triangular array of pebbles arranged in rows with $1, 2, \ldots, n$ pebbles), the squares are of course n^2 (n rows of n pebbles), and the oblong numbers are $n\,(n+1)$ (n rows of $n+1$ pebbles). Some numbers are polygonal in more than one way, for example, 6 is both triangular ($1 + 2 + 3 = T_3$) and oblong ($2 \cdot 3$), while 36 is both square (6^2) and triangular (T_8). Such numbers—the subject of this note—are called *multi-polygonal*.

We begin by presenting a method to generate a sequence containing all the square triangular and oblong triangular numbers. With similar methods we generate sequences with all the pentagonal triangular numbers and all the pentagonal square numbers. Some simple relationships among these numbers are useful in this study. These include $2T_k = k\,(k+1)$ relating triangular and oblong numbers, and $T_k + T_{k+1} = (k+1)^2$ relating triangular and square numbers. Each square k^2 is the arithmetic mean of two consecutive oblong numbers $(k-1)\,k$ and $k(k+1)$, and each oblong number $k(k+1)$ is the geometric mean of two consecutive squares k^2 and $(k+1)^2$. Later we discuss similar relationships involving the pentagonal numbers. For a comprehensive overview of polygonal numbers and their properties, see [**2**].

Square triangular and oblong triangular numbers

The traditional way to find all the square triangular numbers is to solve the equation $n(n+1)/2 = k^2$ for n and k. This equation is equivalent to the Pell equation $x^2 - 2y^2 = 1$, where $x = 2n + 1$ and $y = 2k$. Solving Pell equations is a staple of most elementary number theory textbooks. Similarly, the oblong triangular numbers can be found by solving $n(n+1)/2 = k(k+1)$ for n and k, and this equation is equivalent to the Pell equation $x^2 - 2y^2 = -1$, where $x = 2n + 1$ and $y = 2k + 1$.

Solving Pell equations is hard work, so we take a nontraditional approach and generate a sequence of square and oblong triangular numbers using the following simple theorem relating the factors of one triangular number to a pair of larger triangular numbers with a triangular sum.

Theorem 1. *Let n, p, and q be positive integers. Then*

$$T_n = pq \text{ if and only if } T_{n+p+q} = T_{n+p} + T_{n+q}. \tag{1}$$

Proof. See Figure 1 for a representation of T_{n+p+q} (for $n = 6$, $p = 3$, and $q = 7$).

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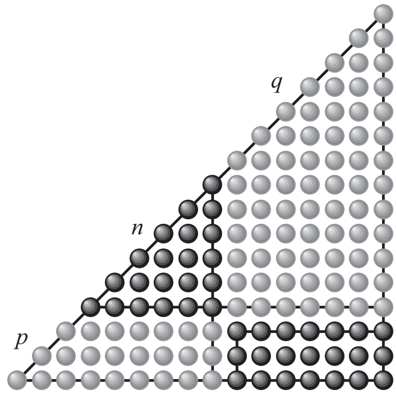

Figure 1

Counting the dots in Figure 1 using the inclusion-exclusion principle yields $T_{n+p+q} = T_{n+p} + T_{n+q} - T_n + pq$, from which (1) follows. The theorem can also be proved algebraically. ■

Figure 1 shows that $T_6 = 21 = 3 \cdot 7$ is equivalent to $T_{16} = T_9 + T_{13}$. Other identities are similarly established, e.g., $T_{5n-2} = T_{3n-1} + T_{4n-2}$ for $n \geqslant 1$ follows from $T_{2n-1} = n(2n-1)$. Two special cases of (1) relate square triangular numbers and oblong triangular numbers:

$$T_n = p^2 \Leftrightarrow T_{n+2p} = 2T_{n+p} = (n+p)(n+p+1) \tag{2a}$$

and

$$T_n = p\,(p+1) \Leftrightarrow T_{n+2p+1} = T_{n+p} + T_{n+p+1} = (n+p+1)^2. \tag{2b}$$

Using (2a) and (2b) repeatedly starting with the smallest square triangular number $T_1 = 1^2$ yields the following sequence of square and oblong triangular numbers:

$$T_1 = 1^2 \Leftrightarrow T_3 = 2T_2 = 2 \cdot 3 = 6,$$

$$T_3 = 2 \cdot 3 \Leftrightarrow T_8 = T_5 + T_6 = 6^2 = 36,$$

$$T_8 = 6^2 \Leftrightarrow T_{20} = 2T_{14} = 14 \cdot 15 = 210,$$

$$T_{20} = 14 \cdot 15 \Leftrightarrow T_{49} = T_{34} + T_{35} = 35^2 = 1225,$$

$$T_{49} = 35^2 \Leftrightarrow T_{119} = 2T_{84} = 84 \cdot 85 = 7140,$$

$$T_{119} = 84 \cdot 85 \Leftrightarrow T_{288} = T_{203} + T_{204} = 204^2 = 41616,$$

$$T_{288} = 204^2 \Leftrightarrow T_{696} = 2T_{492} = 492 \cdot 493 = 242556,$$

$$T_{696} = 492 \cdot 493 \Leftrightarrow T_{1681} = T_{1188} + T_{1189} = 1189^2 = 1413721, \text{ etc.}$$

Let $\mathbb{T}$ denote the sequence of numbers generated by the above procedure, that is, let

$$\mathbb{T} = \{1, 6, 36, 210, 1225, 7140, 41616, 242556, 1413721, \ldots\}.$$

It appears that $\mathbb{T}$ is sequence A096979 in [**5**]. To be precise, $\mathbb{T}$ is the set of triangular numbers such that $T_1 = 1^2 \in \mathbb{T}$ and

$$T_n = p^2 \in \mathbb{T} \Leftrightarrow T_{n+2p} = (n+p)(n+p+1) \in \mathbb{T} \tag{3a}$$

and

$$T_n = p\,(p+1) \in \mathbb{T} \Leftrightarrow T_{n+2p+1} = (n+p+1)^2 \in \mathbb{T}. \tag{3b}$$

Clearly $\mathbb{T}$ contains only square and oblong triangular numbers, and we claim that it contains all of them. Our proof uses the well-ordering principle (every nonempty set of positive integers contains a least element). To begin we "invert" the equivalence (1) by setting $a = n + p$, $b = n + q$, and $c = n + p + q$ (so that $n = a + b - c$, $p = c - b$, and $q = c - a$) to yield

$$T_c = T_a + T_b \Leftrightarrow T_{a+b-c} = (c - a)(c - b). \tag{4}$$

Note that $T_c > T_{a+b-c} \geqslant 1$ because $a + b - c = n \geqslant 1$ and $a + b - c < c$ since $a < c$ and $b < c$. Setting $c = n$ and $a = b = p$ in (4) yields

$$T_n = p\,(p + 1) \Leftrightarrow T_{2p-n} = (n - p)^2, \tag{5a}$$

and setting $c = n$, $a = p$, and $b = p - 1$ in (4) yields

$$T_n = p^2 \Leftrightarrow T_{2p-n-1} = (n - p)(n - p + 1). \tag{5b}$$

Let $\mathbb{T}'$ denote the set of square triangular numbers and oblong triangular numbers that are not elements of $\mathbb{T}$, and assume that $\mathbb{T}'$ is nonempty. By the well-ordering principle, $\mathbb{T}'$ has a least element T_{n_0}. If T_{n_0} is oblong, i.e., if $T_{n_0} = p_0\,(p_0 + 1)$, then from (5a) $T_{2p_0-n_0} = (n_0 - p_0)^2$ is a square triangular number less than T_{n_0}, hence an element of $\mathbb{T}$. If we set $n = 2p_0 - n_0$ and $p = n_0 - p_0$ in (3a), then $n + 2p = n_0$ so that T_{n_0} is an element of $\mathbb{T}$, a contradiction. Hence the least element T_{n_0} is not oblong. Similarly T_{n_0} is not square, so $\mathbb{T}'$ does not have a least element and is thus the empty set. Hence $\mathbb{T}$ contains all the square and oblong triangular numbers.

The structure of the set $\mathbb{T}$ of square and oblong triangular numbers

Combing some of the preceding equivalences reveals some of the structure of $\mathbb{T}$. From (5a) and (3b) we have

$$T_{2p-n} = (n - p)^2 \Leftrightarrow T_n = p\,(p + 1) \Leftrightarrow T_{n+2p+1} = (n + p + 1)^2, \tag{6a}$$

and similarly (5b) and (3a) yield

$$T_{2p-n-1} = (n - p)\,(n - p + 1) \Leftrightarrow T_n = p^2 \Leftrightarrow T_{n+2p} = (n + p)(n + p + 1). \tag{6b}$$

Each oblong triangular number in $\mathbb{T}$ is the geometric mean of the two neighboring square triangular numbers, because (6a) implies that $\sqrt{T_{2p-n}T_{n+2p+1}} = (n - p)\,(n + p + 1) = n\,(n + 1) - p\,(p + 1) = p\,(p + 1) = T_n$ since $n(n + 1) = 2p(p + 1)$. Similarly each square triangular number (greater than 1) in $\mathbb{T}$ is $1/3$ times the arithmetic mean of the two neighboring oblong triangular numbers, because (6b) implies that $\frac{1}{3}\left[\left(T_{2p-n-1} + T_{n+2p}\right)/2\right] = \frac{1}{3}\left(n^2 + n + p^2\right) = p^2 = T_n$ since $n^2 + n = 2p^2$.

The geometric mean and arithmetic mean structure of $\mathbb{T}$ yields a well-known recurrence relation for the square triangular numbers. Let $t_k = a_k^2$ denote the kth square triangular number, e.g., $t_1 = 1^2$, $t_2 = 6^2$, $t_3 = 35^2$, etc., so that $a_1 = 1$, $a_2 = 6$, $a_3 = 35$, etc. Since $t_{k+1} = a_{k+1}^2$ is 1/3 the arithmetic mean of $\sqrt{t_k t_{k+1}} = a_k a_{k+1}$ and $\sqrt{t_{k+1}t_{k+2}} = a_{k+1}a_{k+2}$, we have $a_{k+1}^2 = \frac{1}{6}\,(a_k a_{k+1} + a_{k+1}a_{k+2})$, which simplifies to $a_{k+2} = 6a_{k+1} - a_k$. The sequences $\{a_k\} = \{1, 6, 35, 204, 1189, \ldots\}$, $\left\{a_k^2\right\}$ (the square triangular numbers), and $\{a_k a_{k+1}\}$ (the oblong triangular numbers) appear in [**5**] as sequences A001109, A001110, and A029549, respectively (and $\mathbb{T} = \left\{a_k^2\right\} \cup \{a_k a_{k+1}\}$).

The elements of $\mathbb{T}$ appear in other contexts. For example, in [**1**] Behera and Panda call a positive integer n a *balancing number* if

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$$

for some positive integer r (called the *balancer* of n). In terms of triangular numbers, n is a balancing number if $T_{n-1} = T_{n+r} - T_n$, or equivalently, $T_{n+r} = n^2$. Thus n is a balancing number if and only if its square is in $\mathbb{T}$ (see [**3**] for a visual proof). In [**6**] Panda and Ray call a positive integer m a *cobalancing number* if

$$1 + 2 + \cdots (m-1) + m = (m+1) + (m+2) + \cdots + (m+s)$$

for some positive integer s (called the *cobalancer* of m). In terms of triangular numbers, m is a cobalancing number if $T_m = T_{m+s} - T_m$, or equivalently, $T_{m+s} = m(m+1)$. Thus m is a cobalancing number if and only if $m(m+1)$ is in $\mathbb{T}$. Using (5b) yields $T_{n+r} = n^2 \Leftrightarrow T_{n-r-1} = r(r+1)$, hence the balancer r is a cobalancing number; and similarly, (5a) yields $T_{m+s} = m\,(m+1) \Leftrightarrow T_{m-s} = s^2$, hence the cobalancer s is a balancing number.

Pentagonal triangular numbers

The same procedure we used to construct the sequence of square and oblong triangular numbers can be used to construct the sequence of pentagonal triangular numbers. A pentagonal number enumerates the number of objects in a pentagonal array. In Figure 2(a) we see the pentagonal number P_m (for $m = 4$, $P_4 = 22$). We distort the pentagon to a trapezoid showing that $P_m = T_{2m-1} - T_{m-1}$, while in Figure 2(b) we have $P_m = (1/3)\,T_{3m-1}$, both of which yield $P_m = m\,(3m-1)\,/2$. Other identities relate pentagonal numbers to square and oblong numbers, e.g., $P_m = m^2 + T_{m-1} = m\,(m-1) + T_m$.

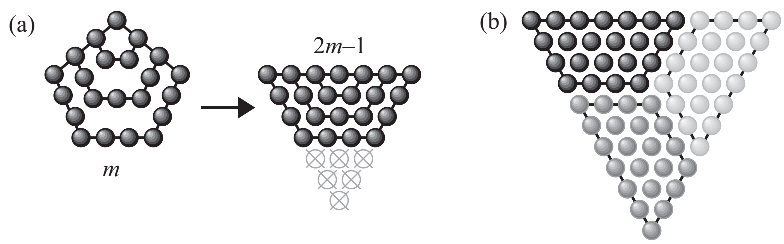

Figure 2

P_m is triangular if there exists a k such that $P_m = T_k$, or equivalently, if there exist a k such that $3T_k = T_n$ where $n = 3m - 1$. In Theorem 2 we show how each solution to $T_n = 3T_k$ yields a larger solution. When $n \equiv -1 (\text{mod } 3)$ we have $P_m = T_k$ for $m = (n+1)/3$.

Theorem 2. *Let n and k be positive integers. Then*

$$T_n = 3T_k \; if\ and\ only\ if \; T_{2n+3k+2} = 3T_{n+2k+1}. \tag{7}$$

Proof. See Figure 3 for a representation of $T_{2n+3k+2}$ (for $n = 3$ and $k = 2$).

Counting the dots in Figure 3 using the inclusion-exclusion principle yields $T_{2n+3k+2} = 3T_{n+2k+1} - 3T_k + T_n$, from which (7) follows. The theorem can also be proved algebraically. ■

Before proceeding, we note that Theorems 1 and 2 are discrete versions of the so-called "carpets theorem": *Place two carpets in room. The area of the carpet overlap equals the area of the uncovered floor if and only if the combined area of the carpets equals the area of the room.*

Defining $T_0 = 0$ and using (7) repeatedly yields the following sequence of results:

$$T_0 = 3T_0 \Leftrightarrow T_2 = 3T_1, \quad T_1 = 1 = P_1,$$

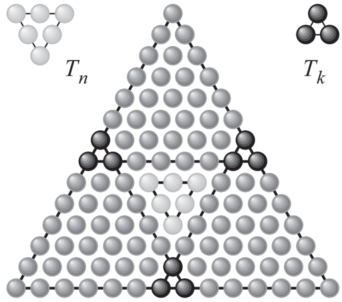

Figure 3

$$\begin{aligned}
T_2 = 3T_1 &\Leftrightarrow T_9 = 3T_5, \quad T_5 = 15,\\
T_9 = 3T_5 &\Leftrightarrow T_{35} = 3T_{20}, \quad T_{20} = 210 = P_{12},\\
T_{35} = 3T_{20} &\Leftrightarrow T_{132} = 3T_{76}, \quad T_{76} = 2926,\\
T_{132} = 3T_{76} &\Leftrightarrow T_{494} = 3T_{285}, \quad T_{285} = 40755 = P_{165},\\
T_{494} = 3T_{285} &\Leftrightarrow T_{1845} = 3T_{1065}, \quad T_{1065} = 567645,\\
T_{1845} = 3T_{1065} &\Leftrightarrow T_{6887} = 3T_{3976}, \quad T_{3976} = 7906276 = P_{2296}, \text{ etc.}
\end{aligned}$$

Let $\mathbb{P}$ denote the sequence of triangular numbers generated by the above procedure, that is, let

$$\mathbb{P} = \{1, 15, 210, 2926, 40755, 567645, 7906276, \ldots\}.$$

It appears that $\mathbb{P}$ is sequence A076139 in [**5**], where it is described as the sequence of triangular numbers that are one-third of another triangular number. To be precise, $\mathbb{P}$ is the set of triangular numbers such that $T_1 = 1 \in \mathbb{P}$ and

$$T_k \in \mathbb{P} \text{ and } 3T_k = T_n \Leftrightarrow T_{n+2k+1} \in \mathbb{P} \text{ and } 3T_{n+2k+1} = T_{2n+3k+2}.$$

The proof that $\mathbb{P}$ contains all triangular numbers that are one-third of another triangular number is analogous to the proof that $\mathbb{T}$ contains every square and oblong triangular number, and is omitted.

An element $T_k = (1/3)\,T_n$ of $\mathbb{P}$ is a pentagonal number P_m when $n \equiv -1 (\text{mod } 3)$, in which case $m = (n+1)/3$. In (7), $n \equiv -1(\text{mod } 3)$ implies $2n + 3k + 2 \equiv 0 \,(\text{mod } 3)$, and $n \equiv 0(\text{mod } 3)$ implies $2n + 3k + 2 \equiv -1 \,(\text{mod } 3)$. Hence every other element of $\mathbb{P}$ is a pentagonal triangular number. Each element of $\mathbb{P}$ that is not pentagonal is a so-called generalized pentagonal number, a positive integer $P_m = m\,(3m-1)\,/2$ where m is a negative integer. For example, $P_{-3} = 15 \in \mathbb{P}$, $P_{-44} = 2926 \in \mathbb{P}$, and $P_{-615} = 567645 \in \mathbb{P}$. In general, an element $T_k = (1/3)\,T_n$ of $\mathbb{P}$ where $n \equiv 0(\text{mod } 3)$ is the generalized pentagonal number $P_{-n/3}$. Thus the elements of $\mathbb{P}$ alternate between pentagonal triangular and generalized pentagonal triangular numbers.

Pentagonal square numbers

The pentagonal number P_m is a square p^2 when $P_m = (1/3)\,T_{3m-1} = p^2$, or equivalently, when $T_{3m-1} = 3p^2$. To find solutions to this equation we employ the next theorem.

Theorem 3. *Let n and k be positive integers. Then*

$$T_n = 3k^2 \text{ if and only if } T_{5n+12k+2} = 3\,(2n+5k+1)^2\,. \tag{8}$$

Proof. Elementary algebra easily shows that $(1/2)\,(5n+12k+2)\,(5n+12k+3) = 3\,(2n+5k+1)^2$ is equivalent to $n(n+1)/2 = 3k^2$.

Using (8) repeatedly starting with $T_0 = 3\cdot 0^2$ yields the following sequence of results:

$$T_0 = 3\cdot 0^2 \Leftrightarrow T_2 = 3\cdot 1^2, \quad P_1 = 1^2,$$

$$T_2 = 3\cdot 1^2 \Leftrightarrow T_{24} = 3\cdot 10^2, \quad P_{-8} = 10^2,$$

$$T_{24} = 3\cdot 10^2 \Leftrightarrow T_{242} = 3\cdot 99^2, \quad P_{81} = 99^2,$$

$$T_{242} = 3\cdot 99^2 \Leftrightarrow T_{2400} = 3\cdot 980^2, \quad P_{-800} = 980^2,$$

$$T_{2400} = 3\cdot 980^2 \Leftrightarrow T_{23762} = 3\cdot 9701^2, \quad P_{7921} = 9701^2, \text{ etc.}$$

Similar to the sequence $\mathbb{P}$ in the preceding section, we have generated a sequence of squares that alternate between pentagonal and generalized pentagonal numbers. The structure of the sequence $\mathbb{S} = \{s_i\} = \{1, 10, 99, 980, 9701, \ldots\}$ of square roots of the pentagonal square and generalized pentagonal square numbers (sequence A004189 in [**5**]) yields a recurrence relation for its elements. Analogous to (6a) and (6b) we have

$$T_{5n-12k+2} = 3(5k-2n-1)^2 \Leftrightarrow T_n = 3k^2 \Leftrightarrow T_{5n+12k+2} = 3(2n+5k+1)^2$$

and $\frac{1}{5}\cdot\frac{1}{2}[(5k-2n-1)+(2n+5k+1)] = k$, so that each element of $\mathbb{S}$ (greater than 1) is 1/5 times the arithmetic mean of its two neighbors. Hence the elements of $\mathbb{S}$ satisfy the recurrence $s_{i+2} = 10s_{i+1} - s_i$.

We conclude by noting that the only pentagonal square triangular number is 1. See [**4, 7**] for details and references.

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Summary. We study some sequences of multi-polygonal numbers, specifically the square triangular, oblong triangular, pentagonal triangular, and pentagonal square numbers. To do so, we state and prove theorems that relate the factors of a given triangular number to a pair of larger triangular numbers with a triangular sum, and relate a triangular number that is three times another triangular number (or three times a square) to a larger pair of triangular numbers with the same property.

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Proof Without Words: Every Cobalancer Is a Balancing Number

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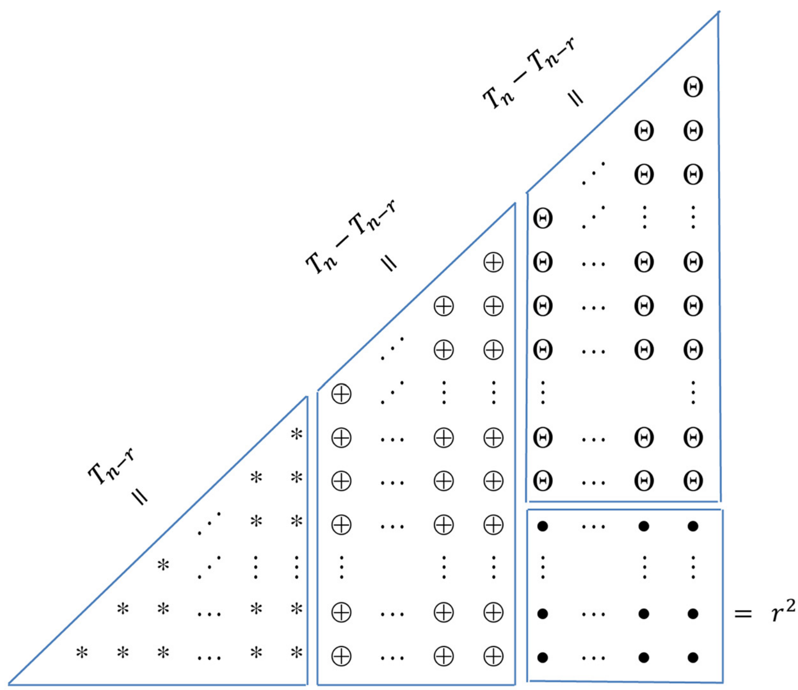
As defined by Panda and Ray [**3**], a cobalancing number (with cobalancer r) is a natural number n satisfying $1 + 2 + \cdots + n = (n + 1) + \cdots + (n + r)$. The cobalancing numbers satisfy the recurrence relation $b_{n+1} = 6b_n - b_{n-1} + 2$ with initial values $b_1 = 0, b_2 = 2$. A natural number n is called a balancing number [**1**] with balancer r if $1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$.

Theorem 1. *A natural number r is a cobalancer (corresponding to a cobalancing number n) if and only if r is a balancing number [**3**].*

Proof.



$$[1 + 2 + \cdots + (n - r)] + [(n - r + 1) + \cdots + n] = [(n + 1) + \cdots + (n + r)]$$

Jones [**2**] showed visually the relationship between balancing numbers and triangular numbers. The following exercise relates balancing and cobalancing numbers.

Exercise. Show the following result. A natural number r is a balancer (corresponding to a balancing number n) if and only if r is a cobalancing number [**3**].

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Summary. A visual proof that every cobalancer is a balancing number.

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Buses, Bullies, and Bijections

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A bully comes onto a bus

In times past in a country far away, passengers on intercity buses were all assigned seat numbers, and the buses were always full. There were many casual people, and there were a few people who were real sticklers for the rules. This is where our problem begins.

Adam is the first person to enter an n seat intercity bus, and, since he is a casual person, he takes a seat at random. Subsequently, $n - 2$ more casual people enter the bus, and they also happily take seats at random. Of course, the inevitable occurs, and the last person on the bus is one of the aforementioned sticklers for the rules.

If luck prevails, the stickler's assigned seat is free; he sits down, and the bus drives away. On the other hand, if his seat is occupied, the stickler insists on having his assigned seat, and he even insists that the person whom he displaces must go to his or her officially assigned seat. Moreover, the stickler continues to oversee the commotion he has created. He insists that each successively displaced person must go to his or her assigned seat until there are no more displaced persons. The $n - 1$ casual people are not happy about this; but, eventually, the brouhaha dies down, and the bus hits the road.

Here is the question: "What is the probability that mellow Adam, the first person on the bus, will be forced to move from his randomly chosen seat?"

If $n = 2$, it is easy to see that the answer is $1/2$. If Adam happens to sit in his assigned seat, then everything is fine; otherwise, he will have to move. Next, we can consider a bus with $n = 3$ seats, but things become more complicated. For a three seat bus, there are $6 = 3!$ possibilities that one needs to consider. To work through these possibilities in a systematic way, one needs some tools like the notation that we develop in the next section. Still, after an examination of the six cases, one comes to a noteworthy observation. In exactly half of these cases, Adam is forced to move.

This coincidence suggests a bold speculation: "Can it be true that the probability that Adam will be forced to move is always one-half whatever the size of the bus may be?"

Representing a permutation in two or more ways

In purely mathematical terms, we have a problem about a randomly chosen permutation of the set $[n] = \{1, 2, \ldots, n\}$, whereby a *permutation* of $[n]$ we just mean a bijection (or one-to-one correspondence) between the elements of the set $[n]$. The prolific

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French mathematician Augustin-Louis Cauchy (1789–1857) suggested that to describe a permutation one should write the elements of $[n]$ in their natural order in one row and then write the values to which they are mapped in a second row. For example, to describe a typical permutation of $[6] = \{1, 2, \dots, 6\}$ one might write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}, \tag{1}$$

by which one would mean that 1 is mapped to 2, and 2 is mapped to 3, and 3 is mapped to 5, and so on. This representation is called the *two-line notation* for the permutation σ, and, after writing a few such representations, one realizes that in most situations the first line can be safely omitted. When one represents a permutation σ with just the second line, we have what is unsurprisingly called the *one-line notation* for a permutation.

There is a third notation for permutations that offer some serious benefits, but its construction calls for a little more thought. The underlying idea is that one takes a value from the set $[n]$, and one then examines where that value is mapped by successive applications of the function $\sigma : [n] \to [n]$. For example, if we take the σ given by the two-lined formula (1) and if we start with 1, then we find the successive images

$$1 \mapsto 2 \mapsto 3 \mapsto 5 \mapsto 1. \tag{2}$$

We could record this information by putting all of the indicated values between parentheses, but that would be a bit redundant since 1 would appear twice. We lose no information if we clip off one of the 1's, and for the moment we will just clip off the last 1. We then record our four element cycle as $(1, 2, 3, 5)$.

We then look at the smallest number that is not in this cycle; in this example that would be 4. To find the cycle containing 4 we again follow the assignments given by Equation (1), and we find $4 \mapsto 6 \mapsto 4$. As before, we drop the repeated value at the end, and we record this cycle information by writing $(4, 6)$. After we have found all of the cycles, we then need to choose an order in which to present them. One natural idea is to order the cycles so that their smallest elements are increasing. In the end, this gives us what we will call the *first* cycle representation of σ,

$$\sigma = (1, 2, 3, 5)(4, 6).$$

Back to the brouhaha on the bus

Returning to the bus, we label the passengers 1 through n, and we assume for each $k \in [n]$ that passenger k is officially assigned seat k. Also, without loss of generality, we assume that the stickler is passenger number 1 and that Adam is passenger number 2. Next, we consider a random permutation $\sigma : [n] \to [n]$, and for $2 \le k \le n$ we let $\sigma(k)$ denote the seat that that passenger k has randomly occupied. By elimination, seat $\sigma(1)$ is the only seat that is empty when the stickler boards the bus.

If $\sigma(1) = 1$ the stickler sits in seat 1, and the bus departs. In such a case, one should note that in the random permutation σ, we have that 1 is in a cycle of length 1. On the other hand, if $\sigma(1) \neq 1$, then the stickler refuses to sit in the empty seat $\sigma(1)$; he wants his assigned seat number 1. This then begins a chain of grumbling dislocations.

To see how the dislocations evolve in a concrete example, we can reuse the sample permutation σ that we used in Equation (1) to illustrate Cauchy's notation. Here we identify an element i of the top row with person i, and, for $2 \le i \le 6$, we let the values in the second row denote the seat numbers that have been occupied at random by the corresponding persons from the top row.

Just before the stickler arrives, only seat 2 is empty; but, when the stickler arrives, he ignores the empty seat. Instead, marches directly to seat 1 where person 5 is sitting. He then makes person 5 move to seat 5. Unfortunately seat 5 is occupied by person 3, so person 3 has to move to seat 3. This seat is occupied by person 2 who then has to move to seat 2. Thankfully seat 2 is empty, so the shuffling can finally end.

The net effect of this drama is that the individuals {2, 3, 5} are the ones who had to move, and the key observation is that these values and {1} are exactly the values that appear in the first cycle of the cycle representation of σ. It's not hard to see that this example is generic; no matter what the size of n or what the random permutation σ, Adam will have to move if and only if Adam is in the first cycle of the cycle representation of σ.

Given this observation, we have a natural plan for finding the probability of Adam having to move. We just need to solve a purely combinatorial problem: "How many permutations of $[n]$ have 1 and 2 in the same cycle?"

A classical counting calculation: Possibly a straw man

Such a natural question is bound to have been asked before, and in fact it has been posed at least once as an exercise (cf. [**9**] p. 58). We will see shortly that there is a lovely bijection that makes it completely obvious that the answer is $n!/2$. Still, why not first give a nod to Kipling and the *Gods of the Copy Book Headings*? In other words, why not be humble and simply do our sums?

Let k be the size of a cycle that contains both 1 and 2. We have $2 \le k \le n$, and there are $\binom{n-2}{k-2}$ ways to select the other elements of $[n] \setminus \{1, 2\}$ that are in the cycle of size k that contains 1 and 2. Given the elements for this cycle of size k, there are then $(k-1)!$ inequivalent ways to order this cycle. Finally, there are $n-k$ elements of $[n]$ remaining, and these can be ordered in $(n-k)!$ ways to make a permutation. By following the first cycle with this permutation, we have defined a one-to-one mapping from $[n]$ onto $[n]$, i.e., we have specified a complete permutation of $[n]$.

Now we do our sums. Specifically, we let Q denote the number of permutations that have 1 and 2 in the same cycle, so, when we condition on $k \in \{2, 3, \dots, n\}$, our earlier observations give us

$$Q = \sum_{k=2}^{n} \binom{n-2}{k-2}(k-1)!(n-k)! = \sum_{k=2}^{n} \frac{(n-2)!}{(n-k)!(k-2)!}(k-1)!(n-k)!$$

$$= (n-2)! \sum_{k=2}^{n}(k-1) = (n-2)! \sum_{k=1}^{n-1} k = (n-2)! \frac{(n-1)n}{2} = \frac{n!}{2}.$$

The charm of this calculation is that it offers good practice with some basic tools of combinatorial counting—plus, of course, it *does* get the job done. Now we know for sure that Adam has to move with probability 1/2—no mater what the size of the bus may be.

On the other hand, just because a problem is solved, it does not mean that one cannot gain more insight. In fact, as promised earlier, there is a simple bijection that recovers the relationships $Q = n!/2$ without the need for any real calculation. Moreover, the same bijection also gives quick answers to more complex problems.

Want insight? Consider a bijection!

The classical calculation got us the answer we hoped to find, but the calculations did not really make us *see* why Adam always has to move with probability one-half. We can do better with a bijection.

As usual, we let $\mathbb{S}_n$ denote the set of permutations of $[n]$, and now we consider a partition of $\mathbb{S}_n$ into two subsets. Specifically, we let $\mathcal{A}_{(1)(2)}$ be the subset of elements of $\mathbb{S}_n$ that have 1 and 2 in different cycles, and we let $\mathcal{A}_{(1,2)}$ be the subset of elements of $\mathbb{S}_n$ that have 1 and 2 in the same cycle. In terms of the cycle representation we can write

$$\mathcal{A}_{(1)(2)} = \{\sigma \in \mathbb{S}_n : (1\ldots)(2\ldots)\ldots\} \quad \text{and}$$
$$\mathcal{A}_{(1,2)} = \{\sigma \in \mathbb{S}_n : (1\ldots 2\ldots)\ldots\}.$$

In each of these formulas the first two ellipses (i.e., the ...'s) can then be any pair of disjoint subsets of $[n] \setminus \{1, 2\}$, including possibly the empty set.

Now consider the transformation $U : \mathcal{A}_{(1)(2)} \to \mathcal{A}_{(1,2)}$ that one gets by erasing the back-to-back parenthesis pair ")(" that precedes the 2 in a permutation $\sigma \in \mathcal{A}_{(1)(2)}$. Thus, for example, if $n = 7$, then one would have

$$\sigma = (153)(24)(67) \mapsto U(\sigma) = (15324)(67) = \tau.$$

To define the inverse transformation $U^{-1} : \mathcal{A}_{(1,2)} \to \mathcal{A}_{(1)(2)}$ we just reverse our recipe; that is, we insert the parenthesis grouping in front of the 2 in a given $\tau \in \mathcal{A}_{(1,2)}$. To continue with our example, one would have

$$\tau = (15324)(67) \mapsto U^{-1}(\tau) = (153)(24)(67) = \sigma.$$

Since we have a bijection between $\mathcal{A}_{(1)(2)}$ and $\mathcal{A}_{(1,2)}$ the two sets have equal cardinalities. Moreover, since their disjoint union is $\mathbb{S}_n$, the sum of these cardinalities is $n!$ and we have

$$|\mathcal{A}_{(1)(2)}| = |\mathcal{A}_{(1,2)}| = \frac{1}{2}n!.$$

This formula seems to provide a richer understanding of the reason why Adam has to move with probability $1/2$. It's because there is a bijection between $\mathcal{A}_{(1)(2)}$ and $\mathcal{A}_{(1,2)}$. Moreover, the bijection is very simple: we just erase the first pair of back-to-back parentheses in the cycle representation of the permutation.

Building a better bijection from $\mathbb{S}_n$ onto $\mathbb{S}_n$

In our continuing example, we began with a permutation σ with Cauchy's representation (1), and we found that there were two cycles that one could write as paths with the same head and tail:

$$1 \mapsto 2 \mapsto 3 \mapsto 5 \mapsto 1 \quad \text{and} \quad 4 \mapsto 6 \mapsto 4. \tag{3}$$

We then built an efficient cycle representation of σ by cutting of the tails of these paths and ordering the resulting strings in increasing order of their heads. The benefit of this representation became evident when it lead us to the surprisingly simple bijection between $\mathcal{A}_{(1)(2)}$ and $\mathcal{A}_{(1,2)}$.

Still, when two paths diverge in a wood, it sometimes pays to go back to see where the other path may have led. Let's consider cutting off the heads in (3) and keeping the tails. We can then order the resulting strings in increasing order of the tails, and we have a *second recipe* for going from a permutation to a uniquely string of parentheses and numbers. For example if $\sigma = [3, 6, 5, 8, 1, 2, 4, 7]$ in one-line notation, then our two recipes give us the strings:

$$\begin{aligned} \text{First Recipe:} \quad & \sigma = (1, 3, 5)(2, 6)(4, 8, 7), \\ \text{Second Recipe:} \quad & \sigma = (3, 5, 1)(6, 2)(8, 7, 4). \end{aligned} \tag{4}$$

When viewed as products of cycles, each of these strings represent the same permutation. Nevertheless, if we ignore the interpretation of these formulas and just look at them as the strings of symbols, then some interesting distinctions emerge. For example, consider a new erasure operation where we erase *all* of the back-to-back parentheses. We then interpret the resulting string as a permutation in one-line notation.

Each of the two recipes then gives us mapping from $\mathbb{S}_n$ to $\mathbb{S}_n$. If we mnemonically call the first mapping F and the second mapping S, then for our example we have

$$\begin{aligned} \text{Using the First Recipe:} \quad & F(\sigma) = [1, 3, 5, 2, 6, 4, 8, 7], \\ \text{Using the Second Recipe:} \quad & S(\sigma) = [3, 5, 1, 6, 2, 8, 7, 4]. \end{aligned}$$

Curiously enough, the mapping $F : \mathbb{S}_n \to \mathbb{S}_n$ has a serious limitation. It is not surjective. To see the problem, just note that for any $\sigma \in \mathbb{S}_n$ the first element of the string $F(\sigma)$ is always equal to 1, so it cannot be a surjection. To be completely explicit just note that $\tau = [2, 3, 4, 5, 6, 7, 1]$ is not equal to $F(\sigma)$ for any σ.

On the other hand, the mapping $S : \mathbb{S}_n \to \mathbb{S}_n$ is an honest bijection—and a stunningly useful one to boot! Since S maps the *finite* set $\mathbb{S}_n$ into itself, the map S must be injective if it is surjective. Thus, to show that S is a bijection, we just need to show that for each $\tau \in \mathbb{S}_n$ there is a $\sigma \in \mathbb{S}_n$ such that $S(\sigma) = \tau$.

We will show this by means of an algorithm. Specifically, we take $\tau \in \mathbb{S}_n$ in its one-line form $[a_1, a_2, \ldots, a_n]$, and we follow a four step process to find a σ such that $S(\sigma) = \tau$.

Step 1: Scan τ until we come to 1, then put down back-to-back parentheses after 1, unless we are at the end—in which case we just stop.

Step 2: Compute the smallest value s that has not yet been scanned.

Step 3: Continue scanning the rest of τ until arriving at s, then put down back-to-back parentheses after s, unless we are at the end—in which case we just stop.

Step 4: Repeat Steps 2 and 3 until done.

To see how the algorithm works, one can take $\tau = [7, 1, 4, 5, 2, 3, 6]$. After Steps 1 and 2, we have the intermediate result $(7, 1)(4, 5, 2, 3, 6)$ and $s = 2$. When we apply Steps 3 and 2 we have $(7, 1)(4, 5, 2), (3, 6)$ and $s = 3$. Finally, after another repetition of Step 3, we have $\sigma = (7, 1)(4, 5, 2)(3)(6)$, and we can safely stop. It is trivial to check that we have the correct result; when we apply the process S of internal parenthesis removal on σ, we have $S(\sigma) = \tau$. Incidentally, this is a good time to note that a cycle of size one is the same as a fixed point, e.g., if 3 is a fixed point of σ, then $\sigma(3) = 3$ and the cycle of σ that contains 3 is simply (3).

At the end of the day we have confirmed that $S : \mathbb{S}_n \leftrightarrow \mathbb{S}_n$ is an honest bijection, but we face some natural questions. Why is this interesting? What can one do with the bijection? We can answer these questions by quizzing ourselves a little harder about Adam and the other boarders of the bus.

Back on the bus with new questions

Suppose that Adam boarded the bus at the same time as four other people from work. What is the probability that the stickler's insistence on the rules will force *all* five of these workers to move from their seats?

In more mathematical terms, we take $2 \le k \le n$, and we ask for the probability that all of the values $\{2, 3, \ldots, k\}$ happen to be in the first cycle of a random permutation of $[n]$. Remarkably, this probability again turns out not to depend on n, and one can see this with help from the bijections S and S^{-1}.

Suppose that $\tau = [a_1, a_2, \ldots, a_n]$ is a random ordering of $[n]$ that we view as the one-line notation of a random permutation τ. One can generate such a random permutation by sampling n times without replacement from the set $[n]$. By the one-to-one correspondence given by S^{-1} this permutation is paired with a unique permutation σ that is given in cycle notation and whose first cycle is $(a_1, a_2, \ldots, a_j)$, where $a_j = 1$ and $1 \le j \le n$.

What's the bottom line? We now see that the probability that the values $\{2, 3, \ldots, k\}$ are all in the same cycle of σ as 1 is the same as the probability that all of the values $\{2, 3, \ldots, k\}$ precede 1 in the one-line permutation $\tau = [a_1, a_2, \ldots, a_n]$. In a random ordering of $[n]$ the values of $[k]$ also appear in random order, so the probability that $\{2, 3, \ldots, k\}$ precede 1 in τ is the same as the probability that 1 is the last number in a random ordering of $[k]$. In such a random ordering, every placement of 1 is equally likely, so the probability that 1 comes last is $1/k$.

Thus, the probability that the values $\{2, 3, \ldots, k\}$ are all in the first cycle of a random permutation of $[n]$ is exactly $1/k$ for all $2 \le k \le n$. For $k = 2$ we get $1/2$, and this is just the result we found from our first bijective argument. For $k = n$, this just reflects that the number of permutations of $[n]$ with 1 in the last position is just $1/n$. To fill in all the values in between, we just use our bijection!

Parenthetically, we should note that the bijection constructed here is essentially the same as a standard ordering that is widely used (e.g., [**6**] or [**1**]). The only distinction is that the more common standard ordering breaks τ at the successive maxima, and, here, for compatibility with our first bijection, we have broken τ at the successive minima.

How many get bumped?

If the set of co-workers on the bus is identified with the set $\{2, 3, \ldots, k\}$ where we have $2 \le k \le n$, then it is also natural to ask about the *total number* of co-workers who get bumped. The bijections S and S^{-1} can help us with this question.

To put the problem formally, we consider the random quantity defined by setting

$$N = \Big|\{j \in \{2, 3, \ldots, k\} \quad \text{such that } j \quad \text{gets bumped}\}\Big|,$$

where we use $|B|$ to denote the cardinality of the finite set B. Here, of course, N is a random variable that depends on the random ordering τ of $[n]$, and N can take on any of the integer values m with $0 \le m \le k - 1$.

We would like to determine the probability mass function of N; that is, we would like to find the probability of the event $\{N = m\}$ for each m. If we reason as before, we have the logical equivalence:

$$N = m \Leftrightarrow \text{ in } \tau \text{ there are exactly } m \text{ elements of } \{2, 3, \ldots, k\} \text{ that precede } 1.$$

Now, in a random ordering of $[k]$, the value 1 occurs with the same probability at each of the k possible places, so the probability that 1 has rank $m + 1$ is the same for all

$0 \le m < k$. Hence, N is uniformly distributed on the set $\{0, 1, \dots, k-1\}$, or, to be explicit, we have

$$P(N = m) = \frac{1}{k} \quad \text{for all } 0 \le m < k. \tag{5}$$

Despite our long familiarity with this result, it still seems remarkable to us that the probability $P(N = m)$ does not depend on m or n except through the minimal requirement that $0 \le m < k \le n$.

As a sidebar, we should also note that (5) implies that the random variable N has expectation

$$E[N] = \sum_{m=0}^{k-1} m \times \frac{1}{k} = \frac{k(k-1)}{2} \times \frac{1}{k} = (k-1)/2.$$

In fact, we could have guessed this from the beginning. The set of co-workers $\{2, 3, \dots, k\}$ has cardinality $k-1$ and each of these co-workers has probability $1/2$ of begin bumped. Linearity of expectation then tells us that the expected number of bumped co-workers has to be $(k-1)/2$, just as one would confirm by calculation from (5).

Noting records: Counting cycles

There is one further topic that is too striking to be left untouched, even if the treatment we give here must be brief. Perhaps the most classical application of the operation of erasing all of the back-to-back parentheses is to determine the mean and variance—or even the distribution—of the number of cycles in a random permutation.

Remarkably enough, it will be useful to introduce the *third* recipe for constructing a cycle representation from the one-line representation of a permutation σ. First, we find the set of cycles as we have done before. Since we can start a cycle with any of the values in the cycle, we now choose to start each cycle with its *largest* value. Finally, we list the cycles in the order that puts leading elements into an *increasing* sequence. For example, if we take our favorite permutation $\sigma = [3, 6, 5, 8, 1, 2, 4, 7]$ in its one-line representation, then our three recipes give us the following strings:

$$\begin{aligned} \text{First Recipe:} \quad & \sigma = (1, 3, 5)(2, 6)(4, 8, 7), \\ \text{Second Recipe:} \quad & \sigma = (3, 5, 1)(6, 2)(8, 7, 4), \\ \text{Third Recipe:} \quad & \sigma = (5, 1, 3)(6, 2)(8, 7, 4). \end{aligned} \tag{6}$$

Here one should note that each of these three strings represents exactly the same permutation σ.

Now, to construct a mappings from $\mathbb{S}_n$ to $\mathbb{S}_n$, we again erase all of the back-to-back parentheses in the third cycle representation. Thus, from the three representations of the permutation σ given above, we find three different images:

$$\begin{aligned} \text{Using the First Recipe:} \quad & F(\sigma) = [1, 3, 5, 2, 6, 4, 8, 7], \\ \text{Using the Second Recipe:} \quad & S(\sigma) = [3, 5, 1, 6, 2, 8, 7, 4], \\ \text{Using the Third Recipe:} \quad & T(\sigma) = [5, 1, 3, 6, 2, 8, 7, 4]. \end{aligned}$$

Finally, one argues as before to show that the third mapping T is really a bijection of $\mathbb{S}_n$ into itself. To underscore the importance of this check, one should recall that S is a bijection, but F is not!

So, how can the new mapping T help us with our declared goal of determining the mean and variance of the number of cycles in a random permutation? The key is that it gives us a way to relate the number of cycles in σ to another notable object, the number of *records* in the image permutation $\tau = T(\sigma)$.

To make this concrete, we introduce random variables R_i, $i = 1, 2, \ldots, n$ that we define by setting $R_i = 1$ or $R_i = 0$ accordingly as a_i is a record maximum or not, as one scans the random permutation (written in one-line notation) $[a_1, a_2, \ldots, a_n]$ from left to right. Naturally we always have $R_1 = 1$ since the first value a_1 is always the largest one that we have seen so far. More generally, if we continue with our familiar example, $\tau = T(\sigma) = [5, 1, 3, 6, 2, 8, 7, 4]$, then we find that R_1, R_4, and R_6 are equal to one, and the rest of the R_i are all equal to zero.

Now we come to a key observation: the number of cycles in the permutation σ is equal to the total number of record maxima in permutation $\tau = T(\sigma)$. Thus, the bijection T shows us that as random variables the number of cycles and the number of record maxima have exactly the same distribution; or, in the language of combinatorics, it shows us that the number of permutations with exactly k cycles is equal to the number of permutations with exactly k records. The punch line is that instead of always working with the number of cycles in a random permutation we can just as well work with the number of records. Moreover, from a probabilistic point of view records turn out to be almost magically nice.

First of all, for all $1 \le k \le n$, we have the notably simple relation

$$E[R_k] = P(R_k = 1) = 1/k.$$

This identity follows from the reasoning we have used before several times. Specifically, if $R_k = 1$ then the kth item in the list has to be the largest among the first k. But the probability that a given element in a list of k elements occupies any particular place in a random ordering of that list (say the last place) is equal to $1/k$.

We noted earlier that the number C_n of cycles of a random permutation σ is equal to the number of maxima that one observes when scanning a random permutation $\tau = T(\sigma) = [a_1, a_2, \ldots, a_n]$, so in symbols we have the identity

$$C_n = R_1 + R_2 + \cdots + R_n.$$

Now, if we take the expectation of both sides of the identity, then by the linearity of expectation (cf. [**2**], p. 206) we find

$$E[C_n] = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \stackrel{\text{def}}{=} H_n \sim \log n \quad \text{as } n \to \infty,$$

and we have the charming appearance of the harmonic number and the logarithm in a problem that begins with counting cycles.

Moreover, one can go much further. The random variables $\{R_i : 1 \le i \le n\}$ are actually independent! At first glance this assertion may seem counterintuitive, but the independence property can be proved without much work; it is a great exercise. Once one has independence, the flood gates of probability theory are open. In particular, we know that for independent random variables, the variance of the sum is equal to the sum of the variances (e.g., [**2**], p. 278), so we ge another harmonic number formula:

$$\text{Var}[C_n] = \sum_{k=1}^{n} \text{Var}[R_k] = \sum_{k=2}^{n} \frac{1}{k}\left(1 - \frac{1}{k}\right) = H_n - \sum_{k=1}^{n} \frac{1}{k^2} \sim \log n \quad \text{as } n \to \infty.$$

Furthermore, by Lindeberg's classic version of the central limit theorem (cf. [**2**], p. 359), one then has for all real values $t \in (-\infty, \infty)$ that

$$\lim_{n\to\infty} P\left(\frac{C_n - \log n}{\sqrt{\log n}} \leq t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx.$$

Among other things, this formula tells us that for large n, one has probability close to 95% that the number of cycles (or the number of records) in a random permutation of $[n]$ is within plus or minus $2\sqrt{\log n}$ of $\log n$. This is surely something that one could not have easily guessed just given the raw first facts about permutations. To be sure, this excursion into the probability of random permutations has gone quickly; but, even such a brief sketch may help to give some sense of the unanticipated power that is cradled in the third recipe.

The rest of the road: What's next?

Our *cri de cour* is simple: "Want insight? Consider a bijection!" This puts one onto a major path of modern combinatorics where one sees many ways that bijections help us to understand combinatorial structures more deeply.

To give a example of a theme worth exploration, one can consider the behavior of cycles in *special subsets of permutations*. For example, consider the set D_n of derangements of $[n]$, i.e., the set of permutations without fixed points. If you choose σ at random from D_n, what is the probability that 1 and 2 are in the same cycle? This is a problem that can be addressed with tools of the kind that we have developed here.

For a more a sophisticated problem with a similar flavor, one can consider the following experiment. Choose independently two cycles of size n, say, for example, $\sigma = (a_1, a_2, \ldots, a_n)$ and $\sigma' = (b_1, b_2, \ldots, b_n)$. Next, consider the permutation defined by the compositional product of these permutations, i.e., consider the mapping $i \mapsto \sigma(\sigma'(i))$. Stanley [**7**] found that the probability p_n that 1 and 2 are in the same cycle of this permutation is given by

$$p_n = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{1}{2} - \frac{2}{(n-1)(n+2)} & \text{if } n \text{ is even.} \end{cases}$$

Thus, for large n, we see that the cycle behavior of the permutations generated by this experiment and the cycle behavior of simple random sampling from $\mathbb{S}_n$ are quite similar—at least with respect to this one particular question. One can anticipate that there are many other sampling experiments for $\mathbb{S}_n$ where analogous behavior can be found.

A little further afield, one of the most famous results in combinatorial probability is Spitzer's identity [**5**], and it also has at its heart a bijection between $\mathbb{S}_n$ and $\mathbb{S}_n$. In fact, simple modifications of the construction of Spitzer and Bohnenblust lead one to a rich family of bijections from $\mathbb{S}_n$ and $\mathbb{S}_n$ (cf. [**8**]).

Finally, we should mention the Robinson–Knuth–Schensted correspondence, which is surely the most notable bijection in the theory of permutations. The RKS correspondence is a bijection between permutations and pairs (P, Q) of remarkable discrete objects called standard Young tableaux. A clear and gentle introduction to the RKS correspondence is given by Stanton and White [**4**], and much further material is given in texts of Bóna [**1**] and Stanley [**6**]. Finally, we should also note that Romik [**3**] gives a wonderful account of the RKS correspondence and its connection to the longest increasing subsequence problem, which is rich topic that has experience stunning progress over recent years.

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Summary. The random—or orderly—seating of passengers on a bus is used to motivate several questions about cycles of permutations. These in turn motivates the investigation of bijections between special subsets of permutations. The goal, of course, is to give simple explanations of surprising facts.

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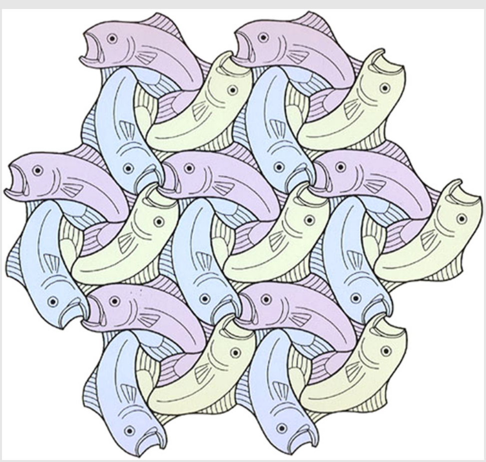
Artist Spotlight
Robert Fathauer



Three Fishes, Robert Fathauer; limited edition screen print, 1994. This work depicts tessellating fish having threefold rotational symmetry about the tail, top fin, and mouth. The motif bears a resemblance to one of Escher's tessellation designs, but the symmetry of the design is quite different. Escher's design has fourfold rotational symmetry about the tail and the top fin, with twofold rotational symmetry about the jaw.

See interview on page 220.

Proof Without Words: Limit of a Recursive Root Mean Square

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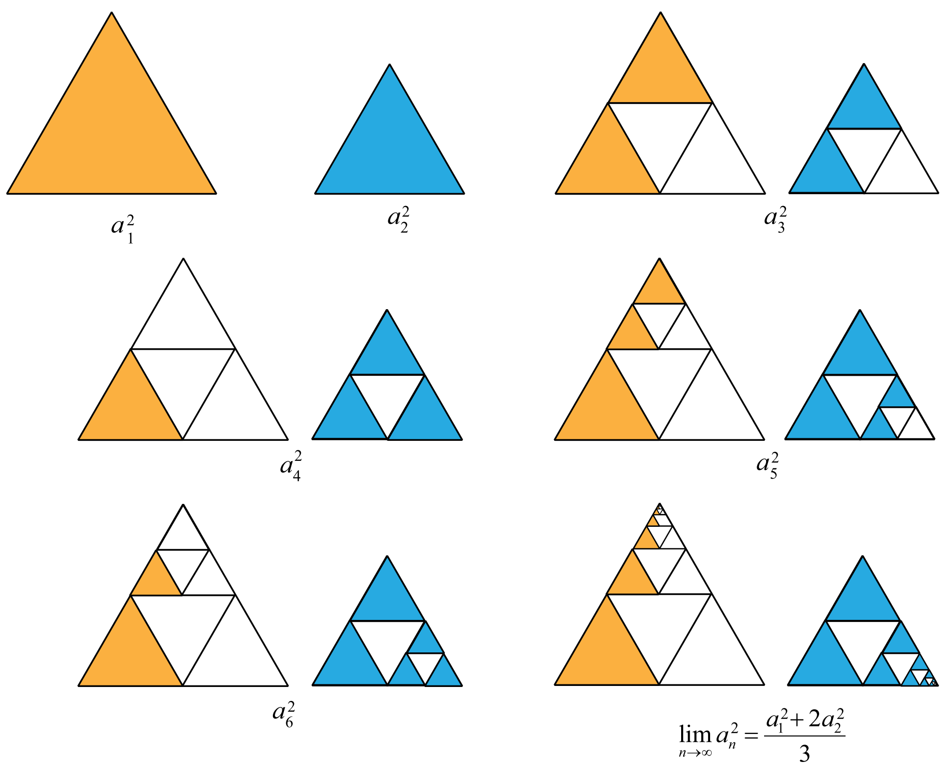
Let $(a_n)_{n\geq 1}$ be the sequence defined recursively by $a_{n+1} = \sqrt{\dfrac{a_n^2 + a_{n-1}^2}{2}}$ for $n \geq 2$, with a_1 and a_2 two positive numbers as initial values. Then $\lim\limits_{n\to\infty} a_n = \sqrt{\dfrac{a_1^2 + 2a_2^2}{3}}$.

Proof.



A related proof without words appears as [1].

REFERENCE

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Math. Mag. **89** (2016) 177–178. doi:10.4169/math.mag.89.3.177.
MSC: Primary 26E60

Summary. Visual proof that the limit of the recursive root mean square sequence defined by $a_{n+1} = \sqrt{\dfrac{a_n^2 + a_{n-1}^2}{2}}$ is $\sqrt{\dfrac{a_1^2 + 2a_2^2}{3}}$ where a_1 and a_2 are the initial values of the sequence.

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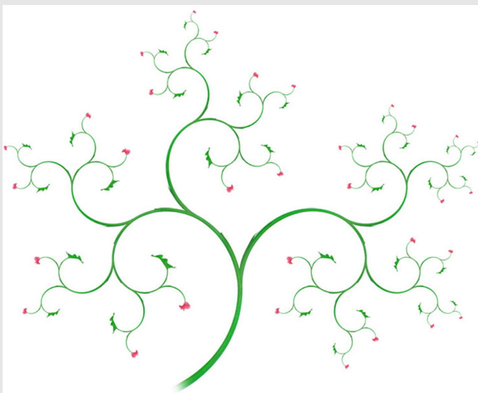
Artist Spotlight
Robert Fathauer



Calla Hibiscus Spirals, Robert Fathauer; limited edition screen print, 2012. This artwork was constructed by graphically iterating a photographic building block created from photographs of a calla lilly and a hibiscus flower, where the stalks of the calla lilly were distorted to conform to plastic-number spirals. In the print, a building block with a large leaf at one end and a blossom at the other was used at the beginning of the iteration process.

See interview on page 220.

Ask Siri . . .

A: What is 1 divided by 0?
S: Please don't make me divide by 0. That would be like asking you to grow a third arm.

$$1 \div 0 = \text{undefined}.$$

Submitted by Anneliese Jones, Ann Arbor, MI

The Coefficients of Cyclotomic Polynomials

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There is a remarkable amount of mathematics to be discovered just by factoring polynomials of the form $x^n - 1$ with $n \in \mathbb{N}$. To get started, consider

$$\begin{aligned} x - 1 &= x - 1 \\ x^2 - 1 &= (x+1)(x-1) \\ x^3 - 1 &= (x^2+x+1)(x-1) \\ x^4 - 1 &= (x^2+1)(x+1)(x-1) \\ x^5 - 1 &= (x^4+x^3+x^2+x+1)(x-1) \\ x^6 - 1 &= (x^2-x+1)(x^2+x+1)(x+1)(x-1). \end{aligned} \tag{1}$$

The polynomials appearing in such factorizations are called **cyclotomic polynomials**. The first few cyclotomic polynomials are

$$\begin{aligned} &\Phi_1(x) = x - 1 \qquad \Phi_2(x) = x + 1 \qquad \Phi_3(x) = x^2 + x + 1 \\ &\Phi_4(x) = x^2 + 1 \qquad \Phi_5(x) = x^4 + x^3 + x^2 + x + 1 \\ &\Phi_6(x) = x^2 - x + 1 \qquad \Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ &\Phi_8(x) = x^4 + 1 \qquad \Phi_9(x) = x^6 + x^3 + 1 \\ &\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1 \\ &\Phi_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ &\Phi_{12}(x) = x^4 - x^2 + 1. \end{aligned}$$

Before giving the official definition of cyclotomic polynomials, we point out some noteworthy patterns that are already apparent among the cyclotomic polynomials listed.

1. It seems that the factors of $x^n - 1$ are exactly those cyclotomic polynomials whose index divides n. For example,

$$x^6 - 1 = \Phi_6(x)\Phi_3(x)\Phi_2(x)\Phi_1(x).$$

2. Looking at $\Phi_2(x)$, $\Phi_3(x)$, $\Phi_5(x)$, $\Phi_7(x)$ and $\Phi_{11}(x)$, it appears that, for prime p,

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1.$$

3. We have $\Phi_4(x) = \Phi_2(x^2)$, $\Phi_8(x) = \Phi_4(x^2) = \Phi_2(x^4)$, $\Phi_9(x) = \Phi_3(x^3)$, and $\Phi_{12}(x) = \Phi_6(x^2)$. (But also $\Phi_6(x) \neq \Phi_3(x^2)$ and $\Phi_6(x) \neq \Phi_2(x^3)$.)

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MSC: Primary 12E10, Secondary 11C08; 13B25.

4. We have $\Phi_6(x) = \Phi_3(-x)$ and $\Phi_{10}(x) = \Phi_5(-x)$. (But also $\Phi_4(x) \neq \Phi_2(-x)$ and $\Phi_{12}(x) \neq \Phi_6(-x)$.)
5. The coefficients of $\Phi_{10}(x)$ put in decreasing degree order are

$$1, -1, \quad 1, -1, \quad 1.$$

Reversing the order of this list leaves it unchanged. Polynomials with this symmetry are called **reciprocal**, and, except for $\Phi_1(x)$, all of the cyclotomic polynomials listed have this property.
6. All coefficients of these cyclotomic polynomials are 0, 1 or -1.

Are these observations about the first 12 cyclotomic polynomials special cases of theorems about all cyclotomic polynomials? As we will see the answer is yes in most cases. Only the last observation (6) about the coefficients of cyclotomic polynomials is wrong in general. It is easy to imagine that the first mathematicians to study these polynomials thought that the coefficients of all cyclotomic polynomials are in $\{-1, 0, 1\}$ because that is indeed the case for $\Phi_n(x)$ with $n < 105$. Remarkably,

$$\begin{aligned} \Phi_{105}(x) = x^{48} &+ x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} \\ &+ x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} - x^{28} - x^{26} \\ &- x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} \\ &+ x^{12} - x^9 - x^8 - 2x^7 - x^6 - x^5 + x^2 + x + 1 \end{aligned} \tag{2}$$

has terms with coefficient -2. This property of Φ_{105} was noted by Migotti [**15**] in 1883 who also proved that, if p and q are distinct odd primes, then the coefficients of Φ_{pq} are in $\{-1, 0, 1\}$ (see Theorem 15). This situation has motivated a large amount of research into the coefficients of cyclotomic polynomials.

Basic properties

To define and to understand cyclotomic polynomials, we need to discuss their zeros. And for that, a bit of group theory—at least the language of group theory—is useful.

The set of nonzero complex numbers, $\mathbb{C}^\times$, forms a group under multiplication. For $\omega \in \mathbb{C}^\times$, the set $\langle\omega\rangle = \{\omega^m \mid m \in \mathbb{Z}\}$ is the **subgroup generated by** ω. The number of elements in this subgroup is called the **order** of ω, which we write as $\operatorname{ord}\omega$. (The usual notation for the order of an element in a group, $|\omega|$, conflicts with the notation for the norm (or absolute value) of complex numbers.)

The connection to the zeros of polynomials of the form $x^n - 1$ is provided by the following group theoretic lemma whose proof can be found in any abstract algebra textbook, for example, [**8**] and [**9**].

Lemma 1. *A complex number $\omega \in \mathbb{C}^\times$ has finite order if and only if $\omega^k = 1$ for some $k \in \mathbb{N}$. If $\operatorname{ord}\omega = n$ is finite, then*

1. *n is the smallest natural number such that $\omega^n = 1$,*
2. *$\langle\omega\rangle = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$,*
3. *for all $m \in \mathbb{Z}$, $\omega^m = 1$ if and only if n divides m.*

There are a lot of easy and useful consequences of this lemma.

1. If $\omega \in \mathbb{C}^\times$ has finite order, then $\omega^{\operatorname{ord}\omega} = 1$.

2. For any $n \in \mathbb{N}$, the zeros of $x^n - 1$ are exactly the complex numbers whose orders divide n.
3. Every complex number of order $n \in \mathbb{N}$ is a zero of $x^n - 1$. So, for example, to find **all** complex numbers of order 4, we need only look among the zeros of $x^4 - 1 = (x^2 + 1)(x + 1)(x - 1)$, namely, 1, -1, i and $-i$. It is easy to check that i and $-i$ are the only complex numbers of order 4, whereas 1 and -1 have orders 1 and 2.

There is other language for describing the complex numbers of interest. For any natural number n, a complex number ω is called an n^{th} **root of unity** if ω is a zero of $x^n - 1$, that is, if $\omega^n = 1$, and ω is called a **primitive** n^{th} **root of unity** if $\operatorname{ord}\omega = n$, equivalently, if ω is a zero of $x^n - 1$ but is not a zero of $x^m - 1$ for any $m < n$.

The key property of the complex numbers is that $x^n - 1$ has exactly n complex zeros, and these can be expressed trigonometrically using De Moivre's theorem.

Lemma 2. *[**9**, p. 18] Let $n \in \mathbb{N}$ and $\omega_n = e^{2\pi i/n} \in \mathbb{C}^\times$. Then $x^n - 1$ has n simple zeros in $\mathbb{C}^\times$, namely,*

$$\omega_n^m = e^{2\pi i m/n} = \cos(2\pi m/n) + i\sin(2\pi m/n)$$

for $0 \le m < n$. Consequently, $\langle\omega_n\rangle$ is the set of zeros of $x^n - 1$ and $\operatorname{ord}\omega_n = n$.

One useful consequence of Lemma 2 is that if two monic polynomials divide $x^n - 1$ for some $n \in \mathbb{N}$, then they are identical if and only if they have the same zeros.

The following lemma provides a formula for the order of ω^m in the case that $\operatorname{ord}\omega$ is known.

Lemma 3. *Suppose that $\omega \in \mathbb{C}^\times$ has finite order. Then, for all $m \in \mathbb{N}$,*

$$m \operatorname{ord}\omega^m = \operatorname{lcm}(m, \operatorname{ord}\omega).$$

Proof. Lemma 1(3) is used four times in this proof! Let $\operatorname{ord}\omega = n$. Because $\omega^{m \operatorname{ord}\omega^m} = (\omega^m)^{\operatorname{ord}\omega^m} = 1$, n divides $m \operatorname{ord}\omega^m$. This implies that $m \operatorname{ord}\omega^m$ is a common multiple of n and m and so $\operatorname{lcm}(m, n)$ divides $m \operatorname{ord}\omega^m$.

On the other hand, because both m and n divide $\operatorname{lcm}(m, n)$, it follows that $\operatorname{lcm}(m, n)/m \in \mathbb{N}$ as well as $(\omega^m)^{\operatorname{lcm}(m,n)/m} = \omega^{\operatorname{lcm}(m,n)} = 1$. Consequently, $\operatorname{ord}\omega^m$ divides $\operatorname{lcm}(m, n)/m$ and, equivalently, $m \operatorname{ord}\omega^m$ divides $\operatorname{lcm}(m, n)$.

Lemma 3 makes it possible to be precise about which complex numbers have finite order.

Lemma 4. *A complex number has order $n \in \mathbb{N}$ if and only if it has the form $\omega_n^m = e^{2\pi i m/n}$ with $0 \le m < n$ and $\gcd(m, n) = 1$.*

Proof. If a complex number has order n, then, by Lemma 1, it is a zero of $x^n - 1$, and, by Lemma 2, has the form ω_n^m for some m such that $0 \le m < n$. Because of Lemma 3, ω_n^m has order n if and only if $\operatorname{lcm}(m, n) = mn$, which is equivalent to $\gcd(m, n) = 1$.

For example, 1, 5, 7, and 11 are the only natural numbers that are less than 12 and relatively prime to 12, and so ω_{12}, ω_{12}^5, ω_{12}^7, and ω_{12}^{11} are the complex numbers of order 12. In general, the number of complex numbers of order n is given by Euler's phi function [**5**] defined by

$$\varphi(n) = \big|\{m \in \mathbb{N} \mid m < n \text{ and } \gcd(m, n) = 1\}\big|.$$

We should mention that all the above results are special cases of theorems that hold in any group. See, for example, [**9**, Section 6] and [**8**, Section 2.3].

We are finally ready to define cyclotomic polynomials.

Definition 5. *For $n \in \mathbb{N}$, the n^{th} **cyclotomic polynomial** is*

$$\Phi_n(x) = \prod_{\operatorname{ord}\omega = n} (x - \omega).$$

Thus, $\Phi_n(x)$ is the monic polynomial whose zeros are the complex numbers of order n.

For example, since 1 is the only the complex number of order 1 and -1 is the only the complex number of order 2 we have

$$\Phi_1(x) = x - 1 \qquad \Phi_2(x) = x + 1.$$

Also, since $\pm i$ are all the complex numbers of order 4 we have

$$\Phi_4(x) = (x - i)(x + i) = x^2 + 1.$$

By Lemma 1(3), the zeros of $x^4 - 1$ are exactly those complex numbers whose orders divide 4. Hence

$$\begin{aligned} x^4 - 1 &= \prod_{(\operatorname{ord}\omega)|4} (x - \omega) \\ &= \left(\prod_{\operatorname{ord}\omega=1} (x - \omega)\right)\left(\prod_{\operatorname{ord}\omega=2} (x - \omega)\right)\left(\prod_{\operatorname{ord}\omega=4} (x - \omega)\right) \\ &= \Phi_1(x)\Phi_2(x)\Phi_4(x). \end{aligned}$$

The argument made for $n = 4$ generalizes very easily to yield the following lemma.

Lemma 6. *For $n \in \mathbb{N}$, $x^n - 1 = \prod_{d|n} \Phi_d(x)$.*

Proof. Since $x^n - 1$ has exactly n simple zeros (Lemma 2), to prove the claim it suffices to check that the polynomials on the left and right of the equal sign have the same zeros. But that is exactly what Lemma 1(3) says.

Lemma 6 makes calculating cyclotomic polynomials much easier. For example, $\Phi_1(x) = x - 1$ and $x^5 - 1 = \Phi_5(x)\Phi_1(x)$, so,

$$\Phi_5(x) = (x^5 - 1)/(x - 1) = x^4 + x^3 + x^2 + x + 1.$$

To calculate $\Phi_{10}(x)$, we use

$$x^{10} - 1 = \Phi_{10}(x)\Phi_5(x)\Phi_2(x)\Phi_1(x).$$

Dividing $x^{10} - 1$ by

$$\Phi_5(x)\Phi_2(x)\Phi_1(x) = (x^4 + x^3 + x^2 + x + 1)(x + 1)(x - 1) = x^6 + x^5 - x - 1$$

we get $\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$. Note that $\Phi_{10}(x) = \Phi_5(-x)$, a relationship that we generalize in Lemma 11.

The fact that all cyclotomic polynomials have integer coefficients is not at all obvious from the definition and needs a proof:

Lemma 7. *For all $n \in \mathbb{N}$, $\Phi_n(x) \in \mathbb{Z}[x]$.*

Proof. We prove the claim by induction on $n \in \mathbb{N}$. Since $\Phi_1(x) = x - 1 \in \mathbb{Z}[x]$ the claim is true for $n = 1$.

Now suppose $n > 1$. By Lemma 6,

$$x^n - 1 = \prod_{d|n} \Phi_d(x) = \Phi_n(x)g(x),$$

where $g(x)$ is the product of all the cyclotomic polynomials $\Phi_d(x)$ with d a proper positive factor of n. By the induction hypothesis, $\Phi_d(x) \in \mathbb{Z}[x]$ for all such cyclotomic polynomials and hence $g(x) \in \mathbb{Z}[x]$. Since cyclotomic polynomials are monic by construction, and products of monic polynomials are monic, $g(x)$ is also monic.

Then $\Phi_n(x)$ is the quotient of $x^n - 1 \in \mathbb{Z}[x]$ by the monic polynomial $g(x) \in \mathbb{Z}[x]$, so $\Phi_n(x)$ is also in $\mathbb{Z}[x]$.

A similar induction proof, left to the reader, shows that $\Phi_n(0) = 1$ for all $n > 1$.

Lemma 8. *For $m, n \in \mathbb{N}$,*

$$\Phi_n(x^m) = \prod_{d \in D} \Phi_d(x),$$

where $D = \{d \in \mathbb{N} \mid \operatorname{lcm}(m, d) = mn\}$.

Proof. Because $\Phi_n(x)$ divides $x^n - 1$, we see that $\Phi_n(x^m)$ divides $x^{mn} - 1$. And, if $d \in D$, then d divides $\operatorname{lcm}(m, d) = mn$ and so, by Lemma 6, the right side of the equation also divides $x^{mn} - 1$. The zeros of $x^{mn} - 1$ are distinct (Lemma 2), so to prove the claim it suffices to confirm that both sides of the equation have the same zeros. For this we just need Lemma 3:

A number $\omega \in \mathbb{C}$ is a zero of $\Phi_n(x^m)$ if and only if $\operatorname{ord} \omega^m = n$, if and only if $\operatorname{lcm}(m, \operatorname{ord} \omega) = mn$, if and only if $\operatorname{ord} \omega \in D$, if and only if ω is a zero of $\prod_{d \in D} \Phi_d(x)$.

For example, if $m = 2$ and $n = 3$, then

$$D = \{d \in \mathbb{N} \mid \operatorname{lcm}(2, d) = 6\} = \{3, 6\}$$

and so $\Phi_3(x^2) = \Phi_6(x)\Phi_3(x)$.

The condition $\operatorname{lcm}(m, d) = mn$ in the definition of D is rather obscure, so to make further use of Lemma 8, we derive a simpler description of D. (See [**5**] for the relevant facts on greatest common divisors and least common multiples.)

Suppose that $d \in D$, that is, $\operatorname{lcm}(m, d) = mn$. Set $k = m/\gcd(m, d)$ so, in particular, $k|m$. Because of the identity $\gcd(a, b) \operatorname{lcm}(a, b) = ab$ for all $a, b \in \mathbb{N}$, we get $n \gcd(d, m) = d$ and $dk = mn$. In addition, because of the identity $a \gcd(b, c) = \gcd(ab, ac)$ for all $a, b, c \in \mathbb{N}$, we have

$$d \gcd(n, k) = \gcd(dn, dk) = \gcd(dn, mn) = n \gcd(d, m) = d$$

and so $\gcd(n, k) = 1$.

Thus, if $d \in D$, then $d = mn/k$ for some $k \in \mathbb{N}$ such that $k|m$ and $\gcd(n, k) = 1$. The converse of this statement can be proved similarly giving

$$D = \left\{ \frac{mn}{k} \mid k \in \mathbb{N} \text{ and } k|m \text{ and } \gcd(n, k) = 1 \right\}. \tag{3}$$

Lemma 9. *If every prime divisor of $m \in \mathbb{N}$ is also a divisor of $n \in \mathbb{N}$, then $\Phi_{mn}(x) = \Phi_n(x^m)$.*

Proof. We use Lemma 8 with D as given by (3). If $d \in D$, then $d = mn/k$ for some $k \in \mathbb{N}$ such that $k|m$ and $\gcd(n, k) = 1$. If p is a prime divisor of k, then, because $k|m$,

p divides m, and then, by assumption, p divides n. But then p divides $\gcd(n,k)$, contradicting $\gcd(n,k) = 1$.

Thus, $k \in \mathbb{N}$ has no prime divisors, $k = 1$ and $d = mn$, $D = \{mn\}$ and $\Phi_n(x^m) = \Phi_{mn}(x)$.

This result enables us to calculate many new cyclotomic polynomials. For example, since $400 = 40 \cdot 10$ and every prime that divides 40 divides 10, we have

$$\Phi_{400}(x) = \Phi_{10}(x^{40}) = x^{160} - x^{120} + x^{80} - x^{40} + 1.$$

Note that $\Phi_{400}(x)$ and $\Phi_{10}(x)$ have the same coefficients.

Corollary 10. *Let n be the product of the prime numbers that divide $m \in \mathbb{N}$. Then $\Phi_m(x) = \Phi_n(x^{m/n})$ and, in particular, $\Phi_m(x)$ and $\Phi_n(x)$ have the same coefficients.*

Proof. Since every prime that divides m/n divides n, this follows directly from Lemma 9.

The main consequence of this corollary is that, for discussion of the coefficients of cyclotomic polynomials, we need only consider $\Phi_n(x)$ when n is a product of distinct prime numbers.

Lemma 11. *If $n \in \mathbb{N}$ is odd, then $\Phi_{2n}(x) = \Phi_n(-x)$.*

Proof. From Lemma 8, we find $\Phi_n(x^2) = \Phi_{2n}(x)\Phi_n(x)$. Replacing x by $-x$ in this equation gives

$$\Phi_{2n}(x)\Phi_n(x) = \Phi_{2n}(-x)\Phi_n(-x). \tag{4}$$

Since $\Phi_n(x^2)$ divides $x^{2n} - 1$, it has only simple zeros. So to prove the claim it suffices to match the zeros on both sides of (4).

If $\Phi_n(\omega) = 0$, then $\operatorname{ord}\omega = n$ so, in particular, $\omega^n = 1$. Since n is odd, $(-\omega)^n = -1$ and so $-\omega$ does not have order n. This means that $-\omega$ must be a zero of $\Phi_{2n}(x)$ and have order $2n$.

Similarly, if $\Phi_{2n}(\omega) = 0$, then $\omega^n \neq 1$ and $(\omega^n)^2 = 1$ and so $\omega^n = -1$. Consequently, $(-\omega)^n = 1$ and so $-\omega$ does not have order $2n$. This means that $-\omega$ has order n and is a zero of $\Phi_n(x)$.

Cyclotomic polynomials have the property that their coefficients are the same when read backward as forward. Such polynomials are called reciprocal polynomials. Specifically, if $f(x)$ is a polynomial of degree m, then $x^m f(1/x)$ is called the **reverse** of f, and f is a **reciprocal polynomial** if it is equal to its reverse, that is, if

$$f(x) = x^m f(1/x). \tag{5}$$

It is not hard to see that the reverse of f is the polynomial f with its coefficients in reverse order. For example, if $f(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$, then

$$\begin{aligned} x^4 f(1/x) &= x^4\left[(1/x)^4 + 2(1/x)^3 + 3(1/x)^2 + 4(1/x) + 5\right] \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 \\ &= 5x^4 + 4x^3 + 3x^2 + 2x + 1. \end{aligned}$$

So a polynomial is reciprocal if and only if the sequence of its coefficients is symmetric with respect to reversal of order. Because of this property, these polynomials are sometimes called **palindromic**.

Lemma 12. *If $n > 1$, then $\Phi_n(x)$ is a reciprocal polynomial.*

Proof. Directly from the definition, if $\omega \in \mathbb{C}^\times$, then $\langle\omega\rangle = \langle\omega^{-1}\rangle$ and so $\operatorname{ord}\omega = n$ if and only if $\operatorname{ord}\omega^{-1} = n$. This means that the function $\omega \mapsto \omega^{-1}$ is a permutation of the set of zeros of $\Phi_n(x)$. Thus, $x^m\Phi_n(1/x)$, with $m = \deg\Phi_n(x)$, has the same set of zeros as $\Phi_n(x)$. The leading coefficient of $x^m\Phi_n(1/x)$ is the constant term of $\Phi_n(x)$ which is 1 for $n > 1$ (see comment after Lemma 7). Thus, $x^m\Phi_n(1/x) = \Phi_n(x)$ for all $n > 1$.

One of the most important properties of cyclotomic polynomials is that they are irreducible over $\mathbb{Q}$. This means that they do not factor into lower-degree polynomials with rational coefficients. Proofs of this fact and its consequences are to be found in many algebra textbooks (which is why this article is focused on the coefficients). See, for example, [**8**, Section 13.6] and [**17**].

The main result

We are now in a position to prove that the coefficients of $\Phi_n(x)$ are in $\{-1, 0, 1\}$ for all $n < 105$. To start, we need some formulas for cyclotomic polynomials whose indices contain two or fewer primes.

Lemma 13. *Let p and q be distinct prime numbers.*

1. $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$
2. $\Phi_q(x^p) = \Phi_{pq}(x)\,\Phi_q(x)$
3. $(x^{pq} - 1)\,\Phi_{pq}(x) = \Phi_q(x^p)\,\Phi_p(x^q)\,(x - 1)$.

Proof. These equations could be obtained from Lemma 8, but it is just as easy to derive them from $\Phi_1(x) = x - 1$ and

$$x^p - 1 = \Phi_p(x)\Phi_1(x) \qquad\qquad x^q - 1 = \Phi_q(x)\Phi_1(x)$$
$$x^{pq} - 1 = \Phi_{pq}(x)\Phi_p(x)\Phi_q(x)\Phi_1(x),$$

which are obtained from Lemma 6.

1. $\Phi_p(x) = (x^p - 1)/(x - 1) = x^{p-1} + x^{p-2} + \cdots + x + 1$.
2. As well as the expression for $x^{pq} - 1$ above, we have

$$\begin{aligned} x^{pq} - 1 = (x^p)^q - 1 &= \Phi_q(x^p)\Phi_1(x^p) = \Phi_q(x^p)(x^p - 1) \\ &= \Phi_q(x^p)\Phi_p(x)\Phi_1(x). \end{aligned}$$

   Cancellation from the two expressions for $x^{pq} - 1$ gives $\Phi_q(x^p) = \Phi_{pq}(x)\,\Phi_q(x)$.
3. This follows from $\Phi_q(x^p) = \Phi_{pq}(x)\,\Phi_q(x)$, $\Phi_p(x^q) = \Phi_{pq}(x)\,\Phi_p(x)$ and the above expression for $x^{pq} - 1$.

We note for future reference that, from (2) or (3) of this lemma, the degree of $\Phi_{pq}(x)$ is $pq - p - q + 1 = (p - 1)(q - 1)$ and is strictly less than pq.

The following lemma is really a weak version of the Chinese remainder theorem [**5**, Theorem 4.8].

Lemma 14. *If p and q are distinct primes, then the coefficients of $\Phi_q(x^p)\,\Phi_p(x^q)$ are in $\{0, 1\}$.*

Proof. From Lemma 13(1), we get

$$\Phi_q(x^p)\,\Phi_p(x^q) = \bigl(1 + x^p + \cdots + x^{(q-1)p}\bigr)\bigl(1 + x^q + \cdots + x^{(p-1)q}\bigr) = \sum_{\substack{0\le m<q\\0\le n<p}} x^{mp+nq}.$$

To complete the proof, it suffices to show that each of the pq terms in this sum has distinct degree. Suppose, to the contrary, that $pm + qn = pm' + qn'$ with $0 \le m < m' < q$. Then $p(m - m') = q(n' - n)$, and, because p and q are distinct primes, q divides $m - m'$. But $0 < m - m' < q$, so this is not possible.

Theorem 15. *If p and q are distinct primes, then the coefficients of $\Phi_{pq}(x)$ are in $\{-1, 0, 1\}$.*

Proof. From Lemma 13(3), we have

$$(x^{pq} - 1)\,\Phi_{pq}(x) = \Phi_q(x^p)\,\Phi_p(x^q)\,(x - 1). \tag{6}$$

Consider the left side of this equation. Since the degree of Φ_{pq} is less than pq, all nonzero terms of $x^{pq}\Phi_{pq}(x)$ have greater degree than the nonzero terms of $\Phi_{pq}(x)$. Hence, the coefficients of $(x^{pq} - 1)\,\Phi_{pq}(x)$ are, up to sign, simply the coefficients of Φ_{pq}.

To complete the proof, it suffices to show that the coefficients on the right side of (6) are in $\{-1, 0, 1\}$. From Lemma 14, the coefficients of $x\Phi_q(x^p)\,\Phi_p(x^q)$ are in $\{0, 1\}$ and the coefficients of $-\Phi_q(x^p)\,\Phi_p(x^q)$ are in $\{0, -1\}$. Hence, the coefficients of the sum of these two polynomials, namely $\Phi_q(x^p)\,\Phi_p(x^q)\,(x - 1)$, are in $\{-1, 0, 1\}$, as claimed.

For another proof, see [**13**].

It is now easy to see that $\Phi_{105}(x)$ is the cyclotomic polynomial of least possible index whose coefficients may not be in $\{-1, 0, 1\}$.

Theorem 16. *If $n \in \mathbb{N}$ has at most two odd prime divisors, then the coefficients of $\Phi_n(x)$ are in $\{-1, 0, 1\}$.*

Proof. We consider several cases:

1. If $n = p$ is prime, then, by Lemma 13(1),

$$\Phi_n(x) = \Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

2. If $n = pq$ for primes p and q, then Theorem 15 applies.
3. If $n = 2pq$ for odd primes p and q, then, by Lemma 11, $\Phi_n(x) = \Phi_{2(pq)}(x) = \Phi_{pq}(-x)$ and Theorem 15 applies again.

Thus, for all these cases, and, by Corollary 10, for all n that have at most two distinct odd prime factors, the coefficients of $\Phi_n(x)$ are in $\{-1, 0, 1\}$.

The smallest $n \in \mathbb{N}$ for which the argument of Theorem 16 fails is the the smallest number that has three distinct odd prime factors, namely $n = 3 \cdot 5 \cdot 7 = 105$. After $n = 105$, the next few numbers that have three or more odd prime factors are $3 \cdot 5 \cdot 11 = 165$, $3 \cdot 5 \cdot 13 = 195$, $2 \cdot 3 \cdot 5 \cdot 7 = 210$, $3 \cdot 7 \cdot 11 = 231$, $3 \cdot 5 \cdot 17 = 255$, $3 \cdot 7 \cdot 13 = 273$, and $3 \cdot 5 \cdot 19 = 285$. Except for $\Phi_{231}(x)$, all of the corresponding cyclotomic polynomials have coefficients that are not in $\{-1, 0, 1\}$.

Other results

Cyclotomic polynomials of the form $\Phi_{pqr}(x)$ with p, q and r odd primes are called ternary. The coefficients of these polynomials continue to be the subject of much research. To discuss this, suppose that $p < q < r$ and let $A(n)$ be the largest coefficient of $\Phi_n(x)$ in absolute value. So, for example, $A(105) = 2$. Then already in 1895, Bang [**3**] proved that $A(pqr) \le p - 1$. In 1968, Beiter [**4**] conjectured that $A(pqr) \le (p+1)/2$ and proved this bound for $p = 3$ and $p = 5$. Much later, it was noticed that $A(17 \cdot 29 \cdot 41) = 10$ whereas, with $p = 17$, $(p+1)/2 = 9$, and so Beiter's conjecture is false. In 2009, Gallot and Moree [**11**] proposed a corrected Beiter conjecture, $A(pqr) \le 2p/3$, that has now been proven by Zhao and Zhang [**18**].

In another direction, G. Bachman [**2**] showed that there are infinitely many ternary cyclotomic polynomials $\Phi_{pqr}(x)$ for which $A(pqr) = 1$. The smallest example of this is $\Phi_{231}(x) = \Phi_{3\cdot 7\cdot 11}(x)$. See also [**12**].

A recent discovery, due to Gallot and Moree [**10**], is that neighboring coefficients of ternary cyclotomic polynomials differ by at most one. This can be seen already in $\Phi_{105}(x)$ (2) whose coefficients, when put in degree order, are

$$1, 1, 1, 0, 0, -1, -1, -2, -1, -1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, -1, 0, -1, 0, -1,$$
$$0, -1, 0, -1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, -1, -1, -2, -1, -1, 0, 0, 1, 1, 1.$$

In particular, -2 in this list is preceded and followed by -1. See also [**6, 7**].

What about cyclotomic polynomials involving four or more primes? In 1931, I. Schur (see [**14, 16**]) showed that there is, in general, no bound on the size of the coefficients of cyclotomic polynomials, essentially because there is no bound on the number of prime numbers. If the goal is to find cyclotomic polynomials with large coefficients, then the obvious candidates are polynomials whose indices are products of many distinct odd primes.

For example, for the product of the first nine odd primes, we get [**1**]

$$A(3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29) = 2\,888\,582\,082\,500\,892\,851.$$

Another product of nine odd primes is

$$N = 13\,162\,764\,615 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 29 \cdot 37 \cdot 43$$

and

$$A(N) = 5\,465\,808\,676\,670\,557\,863\,536\,977\,958\,031\,695\,430\,428\,633.$$

The degree of $\Phi_N(x)$ is $4\,389\,396\,480$, so it is easy to imagine the scale of the computation needed to find $A(N)$. Calculating $A(n)$ when n is a product of 10 distinct primes is still out of the reach of modern computers. For these computational results and much more, see [**1**].

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Summary. One of the most surprising properties of cyclotomic polynomials is that their coefficients are all 1, −1 or zero—at least that seems to be the case until one notices that the 105th cyclotomic polynomial has a coefficient of −2. This article serves as an introduction to these polynomials with a particular emphasis on their coefficients and proves that the coefficients of the first 104 cyclotomic polynomials are at most one in absolute value.

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1 P	2 A	3 T	■	4 G	5 L	6 O	7 M	■	■	8 A	9 D	10 A	11 M	12 S
13 A	N	I	■	14 E	A	S	E	■	15 T	A	E	N	I	A
16 C	O	L	17 U	M	B	U	S	■	18 U	N	F	O	L	D
19 E	D	E	N	■	■	■	20 H	21 O	L	D	E	N	E	R
22 R	E	D	D	23 A	24 W	25 N	■	26 U	S	E	R	■	■	■
■	■	■	■	27 S	H	A	28 S	T	A	■	29 T	30 O	31 P	■
32 A	33 N	34 N	A	L	I	S	A	■	■	■	35 O	P	I	36 E
37 M	A	A	■	■	38 G	A	U	39 G	40 E	■	■	41 A	M	S
42 F	I	Z	43 Z	■	■	■	44 C	A	L	45 C	U	L	U	S
■	46 R	I	O	■	47 P	48 A	E	L	L	A	■	■	■	■
■	■	■	49 M	50 W	A	H	■	51 L	E	N	52 S	53 T	54 R	55 A
56 I	57 Z	58 A	B	E	L	L	59 A	■	■	■	60 T	E	A	M
61 M	E	N	I	A	L	■	62 B	63 E	64 N	65 J	A	M	I	N
66 A	R	T	E	R	Y	■	67 I	P	S	O	■	68 P	S	I
69 M	O	S	S	Y	■	■	70 T	H	A	T	■	71 S	E	O

Proof Without Words: Limit of a Recursive Arithmetic Mean

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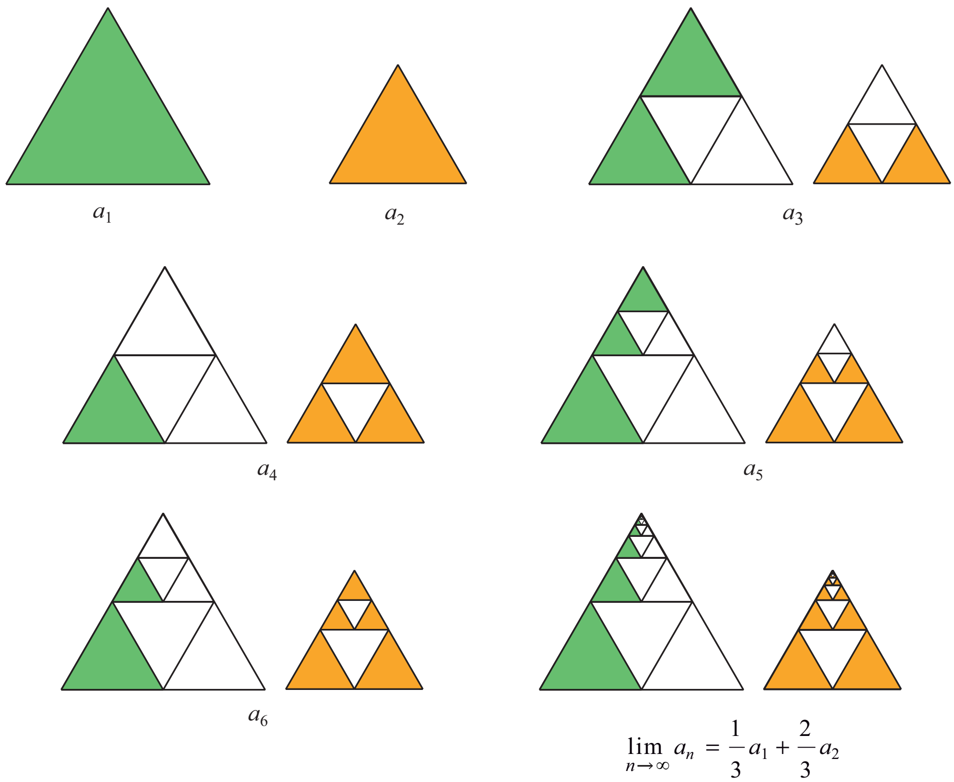
Let $(a_n)_{n\geq 1}$ be the sequence defined recursively by $a_{n+1} = \dfrac{a_n + a_{n-1}}{2}$ for $n \geq 2$, with a_1 and a_2 the initial values. Then $\lim\limits_{n\to\infty} a_n = \dfrac{1}{3}a_1 + \dfrac{2}{3}a_2$.

Proof.



Summary. Visual proof that the limit of the recursive arithmetic mean sequence defined by $a_{n+1} = \frac{a_n+a_{n-1}}{2}$ is $\frac{a_1+2a_2}{3}$, where a_1 and a_2 are the initial values of the sequence.

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A Curious Result for GCDs and LCMs

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A student of mine once pointed out that

$$24 + 36 = \mathrm{LCM}(24, 36) - \mathrm{GCD}(24, 36),$$

where LCM and GCD stand for the least common multiple and greatest common divisor, respectively. Her observation suggests the following question: for which positive integers a and b is $a + b = \mathrm{LCM}(a, b) - \mathrm{GCD}(a, b)$? There is perhaps little reason to believe that such a relationship would ever hold, as the relationship is additive in nature, whereas GCDs and LCMs are multiplicative. Therefore, the following result is a curious one.

THEOREM. *Let a and b be positive integers. Then,*

$$a + b = \mathrm{LCM}(a, b) - \mathrm{GCD}(a, b) \textit{ if and only if } \{a, b\} = \{2k, 3k\}$$

for some positive integer k.

Proof. Suppose a and b are positive integers with

$$a + b = \mathrm{LCM}(a, b) - \mathrm{GCD}(a, b). \tag{1}$$

There exist relatively prime positive integers a' and b' such that $a = a'\mathrm{GCD}(a, b)$ and $b = b'\mathrm{GCD}(a, b)$. Notice that

$$\mathrm{LCM}(a, b) = \mathrm{LCM}(a'\mathrm{GCD}(a, b), b'\mathrm{GCD}(a, b)) = \mathrm{GCD}(a, b)\mathrm{LCM}(a', b').$$

$\mathrm{LCM}(a', b') = a'b'$ as a' and b' are relatively prime. Hence, we may divide both sides of (1) by $\mathrm{GCD}(a, b)$ to obtain $a' + b' = a'b' - 1$. This implies that $a' = \dfrac{b' + 1}{b' - 1}$. However, since a' is a positive integer that is relatively prime to b', one quickly notes that $\{a', b'\} = \{2, 3\}$ and therefore

$$\{a, b\} = \{2(\mathrm{GCD}(a, b)), 3(\mathrm{GCD}(a, b))\} = \{2k, 3k\}, \text{ where } k = \mathrm{GCD}(a, b).$$

Conversely, suppose $\{a, b\} = \{2k, 3k\}$ for some positive integer k. It follows that $\mathrm{GCD}(a, b) = k$ and $\mathrm{LCM}(a, b) = 6k$. Therefore,

$$a + b = 5k = 6k - k = \mathrm{LCM}(a, b) - \mathrm{GCD}(a, b). \qquad \blacksquare$$

Summary. We give necessary and sufficient conditions under which the sum of two positive integers equals the difference of their least common multiple and greatest common divisor, and then prove this result.

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Proof Without Words: Sums of Powers of $\frac{4}{9}$

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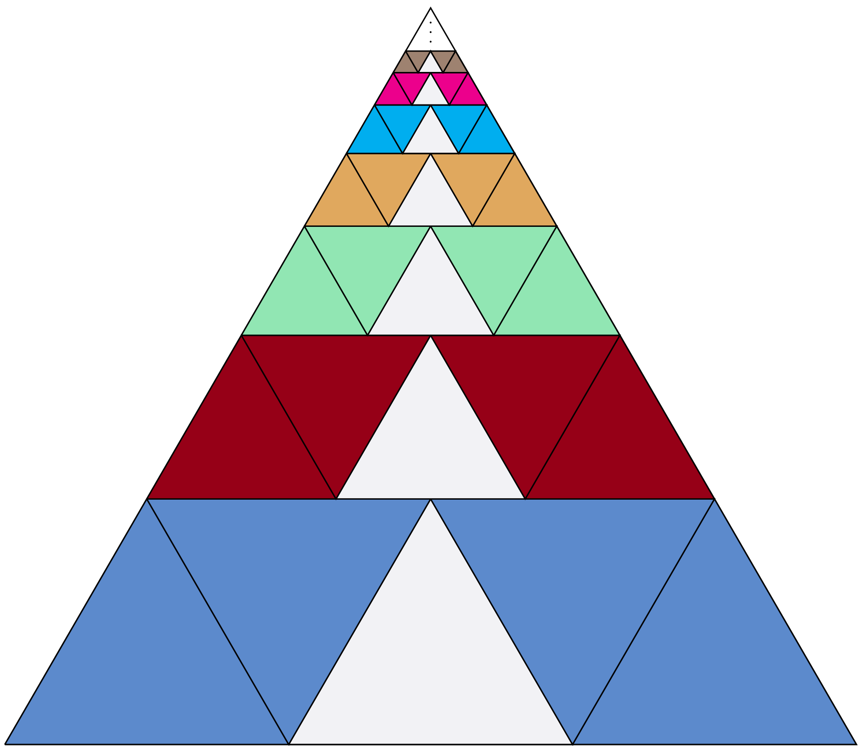
Mabry [2] used a partition of an equilateral triangle into four similar equilateral triangles to show that $\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$. We use a similar approach to show that $\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \cdots = \frac{4}{5}$.



In [1], it is shown that an equilateral triangle can be decomposed into n equilateral subtriangles (as long as $n \neq 2, 3$ or 5); is it possible to determine which other series allow an analogous proof without words?

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Summary. We provide a visual computation of a particular infinite series.

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The Binomial Recurrence

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The solution to the recurrence

$$\left|\begin{matrix} n \\ k \end{matrix}\right| = \left|\begin{matrix} n-1 \\ k \end{matrix}\right| + \left|\begin{matrix} n-1 \\ k-1 \end{matrix}\right| \tag{1}$$

with boundary conditions $\left|\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right| = 1$ and $\left|\begin{smallmatrix} 0 \\ k \end{smallmatrix}\right| = 0$ for $k \neq 0$ is of course the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Such a well-known fact can (not surprisingly) be proved in a variety of ways. One is by substitution of the factorial expressions into the recurrence and verifying that the latter is satisfied. A second uses the combinatorial interpretation of the binomial coefficients (see, for example, Benjamin and Quinn [**1**, p. 64]). These are both standard exercises in introductory combinatorics texts. A somewhat more sophisticated method applies generating functions to derive the binomial theorem, at which point Taylor's formula can be used to show that the solution to the recurrence is $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right| = \frac{n!}{k!(n-k)!}$ (see Wilf [**7**, p. 14]).

There is something a bit unsatisfactory about these techniques, however. The first two methods essentially require one to know (or at least conjecture) the solution to the recurrence in advance. These approaches amount to verifying that the solution is what we already know it is. The third does derive the answer directly from the recurrence, but it invokes the machinery of generating functions and Taylor's formula in order to do so. It seems there should be a method for finding the solution to Equation (1) that does not require that we already know the answer and that only uses basic properties of recurrence relations.

The purpose of this note is to provide such a solution. Specifically, we give an apparently new direct derivation of the solution $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right| = \frac{n!}{k!(n-k)!}$ to Equation (1), using only basic properties of two-variable triangular recurrence relations.

First, some notation. We use the Iverson bracket $[P]$, which evaluates to 1 if statement P is true and 0 if P is false. We consider recurrences of the form

$$\left|\begin{matrix} n \\ k \end{matrix}\right| = f_1(n,k)\left|\begin{matrix} n-1 \\ k \end{matrix}\right| + f_2(n,k)\left|\begin{matrix} n-1 \\ k-1 \end{matrix}\right| + [n=k=0], \text{ for } n,k \geq 0, \tag{2}$$

assuming that $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right| = 0$ when $n < 0$ or $k < 0$. Equation (1) is the case $f_1 = f_2 = 1$.

To derive the solution to Equation (1), we need to establish some properties of recurrence relations of the form of Equation (2). First, it is fairly clear from the recurrence that the only nonzero elements of $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$ occur in the triangle $n \geq k \geq 0$. The second has to do with how changing the f_1 and f_2 functions changes the solution to Equation (2).

Theorem 1. *Suppose*

$$\left|\begin{matrix} n \\ k \end{matrix}\right| = f_1(n,k)\left|\begin{matrix} n-1 \\ k \end{matrix}\right| + f_2(n,k)\left|\begin{matrix} n-1 \\ k-1 \end{matrix}\right| + [n=k=0]$$

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and

$$\left\|{n \atop k}\right\| = h(n)g_1(n-k)f_1(n,k)\left\|{n-1 \atop k}\right\| + h(n)g_2(k)f_2(n,k)\left\|{n-1 \atop k-1}\right\| + [n=k=0].$$

Then the solutions $\left|{n \atop k}\right|$ *and* $\left\|{n \atop k}\right\|$ *are related by*

$$\left\|{n \atop k}\right\| = \left(\prod_{j=1}^{n} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right)\left|{n \atop k}\right|. \tag{3}$$

Proof. The theorem is clearly true in the case $n = k = 0$. Otherwise,

$$\begin{aligned}
&h(n)g_1(n-k)f_1(n,k)\left(\prod_{j=1}^{n-1} h(j)\right)\left(\prod_{j=1}^{n-1-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right)\left|{n-1 \atop k}\right| \\
&\quad + h(n)g_2(k)f_2(n,k)\left(\prod_{j=1}^{n-1} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k-1} g_2(j)\right)\left|{n-1 \atop k-1}\right| \\
&= \left(\prod_{j=1}^{n} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right) f_1(n,k)\left|{n-1 \atop k}\right| \\
&\quad + \left(\prod_{j=1}^{n} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right) f_2(n,k)\left|{n-1 \atop k-1}\right| \\
&= \left(\prod_{j=1}^{n} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right)\left|{n \atop k}\right|.
\end{aligned}$$

Since the right-hand side of Equation (3) satisfies the recurrence for $\left\|{n \atop k}\right\|$, the theorem holds.

Theorem 1 tells us three things:

1. If we multiply f_1 and f_2 by the same function $h(n)$, the solution to the recurrence is multiplied by a factor of $\prod_{j=1}^{n} h(j)$.
2. If we multiply f_1 by the function $g_1(n-k)$, the solution to the recurrence is multiplied by a factor of $\prod_{j=1}^{n-k} g_1(j)$.
3. If we multiply f_2 by the function $g_2(k)$, the solution to the recurrence is multiplied by a factor of $\prod_{j=1}^{k} g_2(j)$.

Since we can use Theorem 1 to create solutions to recurrences of the form (2) from known solutions, a natural question to ask is this: What are the simplest (in some sense) functions f_1 and f_2 that give rise to $\left|{n \atop k}\right| = 1$ for $n \geq k \geq 0$ (i.e., yield a triangle of 1's)?

Let's start with the boundaries of the triangle. Since $\left|{n \atop -1}\right| = 0$, when generating the column $\left|{n \atop 0}\right| = 1$ the recurrence simplifies to $\left|{n \atop 0}\right| = \left|{n-1 \atop 0}\right|$. This implies that $f_1(n, 0) = 1$. Similarly, since $\left|{n-1 \atop n}\right| = 0$, to generate the diagonal $\left|{n \atop n}\right| = 1$, the recurrence simplifies to $\left|{n \atop n}\right| = \left|{n-1 \atop n-1}\right|$, and thus $f_2(n, n) = 1$. Off of the boundaries, substituting 1's into $\left|{n \atop k}\right|$, $\left|{n-1 \atop k}\right|$, and $\left|{n-1 \atop k-1}\right|$, we get that $f_1(n, k) + f_2(n, k) = 1$. This rules out constant solutions for f_1 and f_2. The simplest solutions that depend on n and k and that yield $f_2(n, n) = 1$

are probably $f_2(n,k) = \frac{n}{k}$ and $f_2(n,k) = \frac{k}{n}$. The former leads to $f_1(n,k) = 1 - \frac{n}{k}$, which is undefined when $k = 0$ and thus is ruled out. However, the latter leads to $f_1(n,k) = 1 - \frac{k}{n}$, which satisfies the other boundary condition $f_1(n,0) = 1$. We thus have our third property of recurrences of the form of Equation (2), which we might as well call a theorem:

Theorem 2. *If $f_1(n,k) = 1 - \frac{k}{n}$ and $f_2(n,k) = \frac{k}{n}$, then the solution to Equation* (2) *is $\left|{n \atop k}\right| = 1$ for $n \geq k \geq 0$.*

(The functions f_1 and f_2 in Theorem 2 have the additional nice property that $f_1(n,n) = 0$ and $f_2(n,0) = 0$. Thus, even if the definition did not force the values off of the triangle $n \geq k \geq 0$ to be zero, these off-values would still not affect the values inside the triangle.)

With Theorems 1 and 2 in place, the proof of our main result is straightforward.

Theorem 3. *The solution to*

$$\left|{n \atop k}\right| = \left|{n-1 \atop k}\right| + \left|{n-1 \atop k-1}\right| + [n = k = 0]$$

is

$$\left|{n \atop k}\right| = \frac{n!}{k!\,(n-k)!}.$$

Proof. Starting with $f_1(n,k) = 1 - \frac{k}{n}$ and $f_2(n,k) = \frac{k}{n}$ from Theorem 2, let's find functions h, g_1, and g_2 such that $h(n)g_1(n-k)f_1(n,k) = h(n)g_2(k)f_2(n,k) = 1$. We need to clear the denominators of f_1 and f_2, and so we need $h(n) = n$. This leaves $g_1(n-k)(n-k) = 1$, and thus $g_1(n-k) = \frac{1}{n-k}$. Similarly, we have $g_2(k)k = 1$, and thus $g_2(k) = \frac{1}{k}$. By Theorem 1, we thus have that the solution is

$$\left|{n \atop k}\right| = \left(\prod_{j=1}^{n} j\right)\left(\prod_{j=1}^{n-k} \frac{1}{j}\right)\left(\prod_{j=1}^{k} \frac{1}{j}\right) = \frac{n!}{(n-k)!\,k!}.$$

Recurrence relations of the form of Equation (2) have generally been difficult to solve, even though many important named numbers are special cases. (Besides the binomial coefficients, different forms of f_1 and f_2 generate both kinds of Stirling and associated Stirling numbers, the Lah numbers, the Gaussian coefficients, the Eulerian numbers, and second-order Eulerian numbers. See Konvalina [**4**] for a unified combinatorial interpretation of some of these numbers.) In fact, Research Problem 6.94 in Graham, Knuth, and Patashnik's *Concrete Mathematics* [**3**] says

> Develop a general theory of the solutions to the two-parameter recurrence
>
> $$\left|{n \atop k}\right| = (\alpha n + \beta k + \gamma)\left|{n-1 \atop k}\right| + (\alpha' n + \beta' k + \gamma')\left|{n-1 \atop k-1}\right| + [n = k = 0], \quad \text{for } n, k \geq 0, \quad (4)$$
>
> assuming that $\left|{n \atop k}\right| = 0$ when $n < 0$ or $k < 0$. What special values $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ yield "fundamental solutions" in terms of which the general solution can be expressed?

Much recent progress has been made on this problem, however. In previous work [**6**], we use our Theorem 1 and some similar results to find explicit solutions to several cases of Equation (4). In addition, Mansour and Shattuck [**5**] give combinatorial interpretations of several cases of Equation (4). Finally, a recent article by Barbero, Salas, and Villaseñor [**2**] gives a complete solution to Equation (4) in terms of generating functions.

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Summary. We give a new, direct argument that the solution to the binomial recurrence is the binomial coefficient. Our argument does not assume that the solution is known in advance nor does it rely on anything other than basic properties of two-variable triangular recurrence relations.

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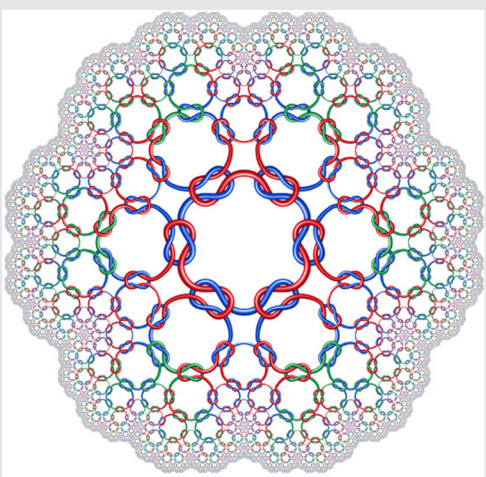
Artist Spotlight
Robert Fathauer



Square Knots Stretching to Infinity, Robert Fathauer; 14 in. × 14 in. limited edition of 50 digital prints, 2007. This print depicts a fractal link design where each strand has four lobes and connects to other strands via square knots. Three colors are required to avoid knotted strands having the same color. The design was created by decorating the individual tiles in a fractal tiling with knot-like graphics.

See interview on page 220.

Proof Without Words: Sums of Consecutive Odds and Positive Integer Cubes

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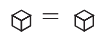
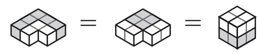
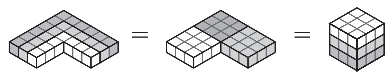
$1 = 1^3$



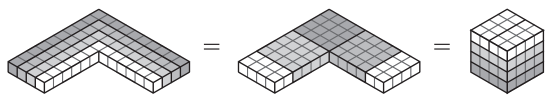
$3 + 5 = 2^3$



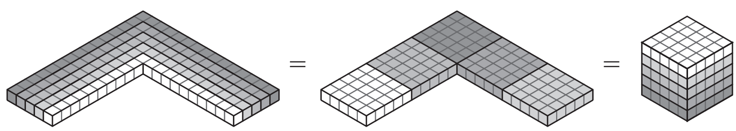
$7 + 9 + 11 = 3^3$



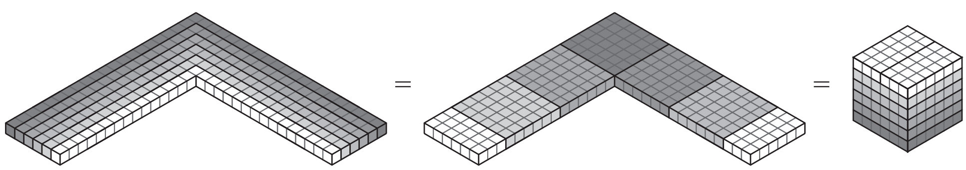
$13 + 15 + 17 + 19 = 4^3$



$21 + 23 + 25 + 27 + 29 = 5^3$



$31 + 33 + 35 + 37 + 39 + 41 = 6^3$



$$\vdots$$

$$[n(n-1)+1] + [n(n-1)+3] + \cdots + [n(n+1)-1] = n^3$$

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An Algebraic Method to Find the Symmedian Point of a Triangle

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The *symmedian point* does not get much attention despite being branded "one of the crown jewels of modern geometry." [**2**, Chapter 7]. Surprisingly, many students, mathematics educators, and mathematicians have never heard of it. In [**5**] the author collects an extensive history of this "remarkable point." He discovers that the first mention of the symmedian point was in connection with two important facts. First, the distances of the symmedian point from the sides of a triangle are directly proportional to the sides. Second, the point in a triangle from which perpendiculars are drawn to the sides of the triangle so that the sum of their squares is the least possible is the symmedian point. More recently, Ross Honsberger devotes more than 25 pages to it [**2**, Chapter 7], and R. K. Smither [**7**] gives an application of the point when analyzing data from a test of a harbor mine-detecting algorithm. He calls this point *the least-squares point*. He mentions that the least-squares point and the symmedian point are the same (for a proof of this fact see [**1**, item 14]). Then he provides a geometric way of constructing the least-squares point.

In this paper, we first recall the definition of the symmedian point for a triangle. Then we set up a natural least-squares problem for which the symmedian point is the solution. The exact coordinates of the least-squares point (hence the symmdian point) will also be given. This provides an interesting link between the least-squares method and geometry. A natural extension of this result to higher dimensions is mentioned at the end of the paper.

We start with the definition of the symmedian point. In a triangle ABC let AL and AM be the angle bisector of $\angle A$ and the median of the side BC, respectively (see Figure 1). Let AN be obtained by reflecting AM across AL. The line through AN is called *symmedian* (an abbreviation for "symmetry of the median"). It turns out that the symmedians through the vertices A, B, and C are concurrent. The intersection point K is called the *symmedian point* of the triangle ABC (see Figure 1). In Britain and France this point is known as the Lemoine point, and in Germany it is called Grebe's point; for example, see [**3**, pp. 213–218] and [**2**, Chapter 7].

Now we set up a least-squares problem for which the symmedian point is the solution. Suppose that three lines L_1, L_2, and L_3 form a triangle in the plane (see Figure 2). We write

$$L_i : a_i x + b_i y = c_i \quad \text{for} \quad i = 1, 2, 3,$$

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MSC: Primary 51A05, Secondary 15A06

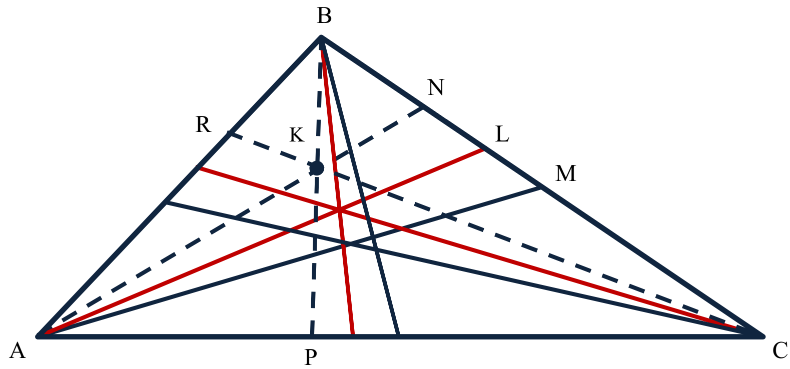

Figure 1 The line segments AN, BP, and CR that are generated by reflecting the medians of triangle ABC across the appropriate angle bisectors are called symmedians. The symmedian lines of a triangle always intersect at a unique point known as the symmedian point, denoted by K.

where we may assume that the linear equations are normalized, that is, $a_i^2 + b_i^2 = 1$ for $i = 1, 2, 3$. Consider the least-squares problem

$$Aw = c, \tag{1}$$

where

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}, \quad w = \begin{bmatrix} x \\ y \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Recall that if $L : ax + by = c$ with $a^2 + b^2 = 1$, then the square of the distance between the point $P(x, y)$ and the line L is given by

$$\left[\frac{|ax + by - c|}{\sqrt{a^2 + b^2}}\right]^2 = \frac{(ax + by - c)^2}{a^2 + b^2} = (ax + by - c)^2. \tag{2}$$

Now the least-squares solution $(\hat{x}, \hat{y})$ of (1) is given by

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \hat{X} = (A^T A)^{-1} A^T c. \tag{3}$$

In addition, $\|A\hat{X} - c\| \le \|AX - c\|$ for all $X \in R^2$, where $\|\cdot\|$ denotes the standard Euclidean norm (see [**4**, Chapter 6]). It follows that

$$\sum_{i=1}^{3} (a_i \hat{x} + b_i \hat{y} - c_i)^2 \le \sum_{i=1}^{3} (a_i x + b_i y - c_i)^2$$

for all (x, y). Thus, in view of (2), the least-squares solution $(\hat{x}, \hat{y})$ of (1) is the least-squares point (and therefore the symmedian point) of the triangle determined by the three lines L_1, L_2, and L_3. Moreover the symmedian point $(\hat{x}, \hat{y})$ coincides with the centroid of the pedal triangle A′B′C′, which is the triangle obtained by projecting $(\hat{x}, \hat{y})$ onto the sides of triangle ABC (see Figure 2). By solving (3), we obtain explicit formulas for the coordinates of the least-squares point,

$$\hat{x} = \frac{(\sum_{i=1}^{3} b_i^2)(\sum_{i=1}^{3} a_i c_i) - (\sum_{i=1}^{3} a_i b_i)(\sum_{i=1}^{3} b_i c_i)}{(\sum_{i=1}^{3} a_i^2)(\sum_{i=1}^{3} b_i^2) - (\sum_{i=1}^{3} a_i b_i)^2}$$

and

$$\hat{y} = \frac{(\sum_{i=1}^{3} a_i^2)(\sum_{i=1}^{3} b_i c_i) - (\sum_{i=1}^{3} a_i b_i)(\sum_{i=1}^{3} a_i c_i)}{(\sum_{i=1}^{3} a_i^2)(\sum_{i=1}^{3} b_i^2) - (\sum_{i=1}^{3} a_i b_i)^2}.$$

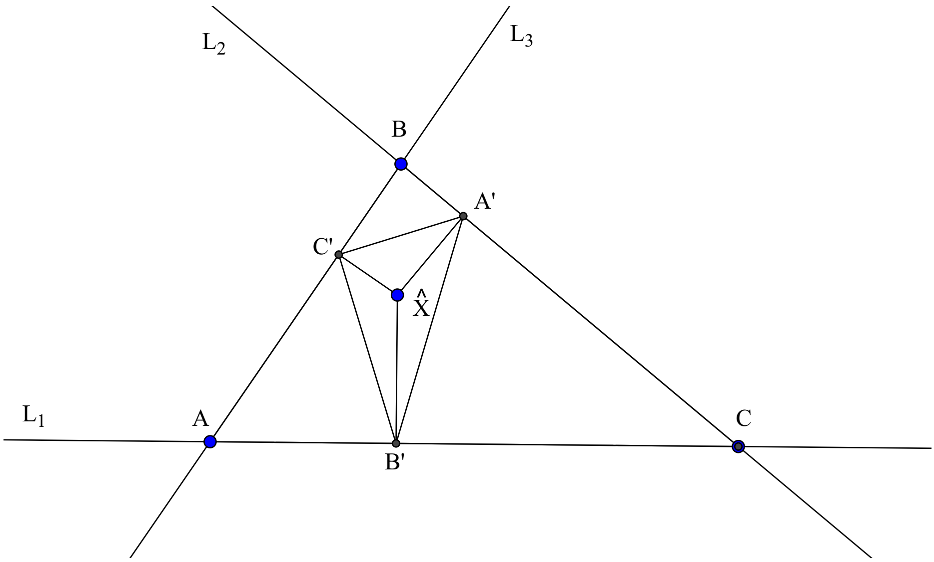

Figure 2 The intersections of lines L_1, L_2, and L_3 form the triangle ABC. The least-squares solution $\hat{X}$ of (1) coincides with the symmedian point of triangle ABC and the centroid of the pedal triangle A′B′C′.

Finally, we make two remarks. First, the algebraic method presented in this paper can be naturally generalized to higher dimensions. Specifically, let H be a set of m hyperplanes in $\mathbb{R}^n$ defined by

$$H = \left\{ \sum_{j=1}^{n} a_{ij} x_j = c_i \,\middle|\, a_{ij}, c_i, x_j \in \mathbb{R}, 1 \le j \le n, 1 \le i \le m \right\}, \tag{4}$$

where the associated half-planes form a closed and bounded region D. Again, we assume the equations of the hyperplanes are normalized. The set H is represented by the linear system $AX = b$, where $A = [a_{ij}]$ is $m \times n$ and $b = [c_i]$. An argument identical to the one used above, for the case of a triangle, shows that the least-squares point of the compact region D defined in (4) and the least-squares solution of $AX = b$ coincide. Second, the authors provided a similar definition of the symmedian point for a tetrahedron whereby the least-squares point of the tetrahedron and the symmedian point coincide [**6**].

Acknowledgment The authors would like to thank the referees and the editor for their valuable suggestions that have improved this article.

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Summary. A relationship between the symmedian point of a triangle and the least-squares solution of a linear system is presented. The coordinates of the symmedian point are explicitly calculated as a solution to the linear system.

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Ask Siri . . .

A: What is 2 divided by 0?
S: You know, that's not nice. Are you trying to scramble my computational cortex?

$$2 \div 0 = \text{undefined}.$$

Submitted by Anneliese Jones, Ann Arbor, MI

Building the Biggest Box: Three-factor Polynomials and a Diophantine Equation

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As every instructor knows, and as most students come to realize, optimization problems are the bread and butter of introductory calculus. The following problem is ubiquitous.

> A box with an open top is to be constructed from a square piece of cardboard, 3 ft. wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have. [**18**, Section 3.7]

From a pedagogical point of view this problem is nice since it requires optimizing a third degree polynomial: the first derivative has two critical points and a careful student will have to justify which one yields the solution to the problem.

Thinking pragmatically (e.g., with an eye to grading an exam question), the problem as stated above has the added benefit that the dimensions of the box with largest volume are rational. This leads to the following question: if we take the sheet of cardboard to be a rectangle, what other integer dimensions yield an answer which is rational? As the second author discovered (under the pressure of setting a make-up exam), it is not easy to find such examples, though they do exist. In this paper we give a complete answer to this question.

Theorem 1. *Let u and v be natural numbers with $u \geq v$ and $\gcd(u, v) = 1$. Then the equation*

$$\frac{A}{B} = \frac{1}{2} + \frac{3u^2 - v^2}{4uv}$$

gives all ratios $\frac{A}{B}$ such that a rectangle with integer side lengths A and B, $A \geq B$, will yield a rational solution to the box problem.

While the problem of rational solutions has been studied before (see the last section), we have not found this generating formula anywhere in the literature. Surprisingly, the proof of Theorem 1 reduces to a problem in elementary number theory: find all the solutions of the Diophantine equation

$$a^2 + 3b^2 = c^2, \quad a, b, c \in \mathbb{N}.$$

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This equation resembles the more famous Pythagorean equation

$$a^2 + b^2 = c^2,$$

and our proof is based on one of the many approaches to finding the integer solutions of this equation. The factor of 3, however, introduces some interesting difficulties into the proof.

However, rather than concentrating on the box problem, we want to consider a more general calculus problem that in turn leads to a more general Diophantine equation. We have not encountered this calculus problem before, and we hope that it will be a small but useful addition to the calculus repertoire.

To motivate this generalization, we start with the calculus part of the box problem. Given a cardboard rectangle with dimensions A and B, $A \geq B > 0$, maximizing the volume of the open top box requires finding the local maxima of the third degree polynomial

$$V = x(2x - A)(2x - B), \quad 0 < x < B/2.$$

Since maximizing V is the same as maximizing $2V$, we can replace $2x$ by x and consider instead the polynomial

$$y = x(x - A)(x - B). \tag{1}$$

The box problem imposes the natural constraint that A, B are positive, but in the abstract we can take $A,\ B \in \mathbb{Z}$.

We generalize this cubic polynomial by replacing x with x^m, $m \in \mathbb{N}$:

$$y = x(x^m - A)(x^m - B) = x^{2m+1} - (A + B)x^{m+1} + ABx, \quad A,\ B \in \mathbb{Z}. \tag{2}$$

By a small abuse of terminology we will call these three-factor polynomials. Geometrically, the graphs of these polynomials have some interesting properties depending on the parity of m and the signs of A and B. For example, suppose $m = 2$. If $A,\ B > 0$, the graph has five real roots; if $B < 0 < A$ then the graph has three real roots and three consecutive inflection points. See Figure 1.

When $m > 1$ this equation need not have rational roots, but its roots will all be "nice" in the sense that they are mth roots of integers; for brevity we call these m-rationals. We generalize the box problem by asking which values of A and B are such that the first and second derivatives of this polynomial also have m-rational roots. (In the box problem, the second derivative is linear and so always has rational roots.) The following theorem completely characterizes these values.

Theorem 2. *Given $m \in \mathbb{N}$, let u and v be natural numbers such that* $\gcd(u, v) = 1$ *and $u \geq \frac{v}{\sqrt{2m+1}}$. Then the equation*

$$\frac{A}{B} = \frac{1 + 2m - m^2}{(m+1)^2} \pm \frac{2m}{(m+1)^2}\sqrt{\left(\frac{(2m+1)u^2 + v^2}{2uv}\right)^2 - (2m+1)} \tag{3}$$

gives all ratios $\frac{A}{B}$, $A,\ B \in \mathbb{Z}$, such that the zeros of the first and second derivatives of the polynomial

$$y = x(x^m - A)(x^m - B)$$

are m-rational.

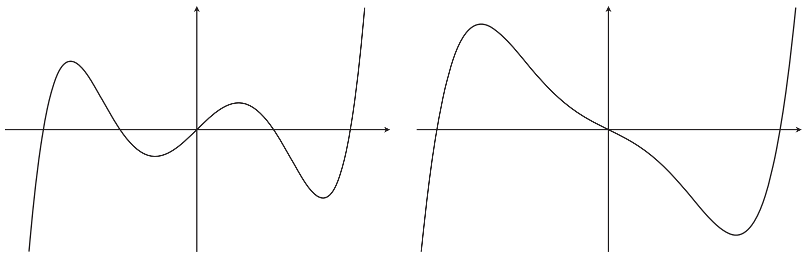

Figure 1 Graphs of three-factor polynomials with $m = 2$ (i.e., degree 5). The graph on the left has five distinct zeros and four local extrema. The graph on the right has three consecutive inflection points with no local extrema between.

The proof of Theorem 2 reduces to the solution of another Diophantine equation,

$$a^2 + kb^2 = c^2, \quad k,\ a,\ b,\ c \in \mathbb{N}, \tag{4}$$

and again the solution is similar to that of the classical Pythagorean equation. The interesting new facets of the proof come when we consider the role played by the prime factors of k.

In the remainder of this paper we first show how the problem of finding m-rational roots for a three-factor polynomial and its derivatives reduces to finding the solutions of (4). We will then derive the complete solution to this equation; for clarity we split the solution into three parts, which we give as Theorems 3, 4, and 5. We then work backwards to prove Theorems 1 and 2. Finally, we will briefly survey the history of the box problem and the underlying Diophantine equations. Both have been considered multiple times in the literature, though our solution employs some elegant number theory, especially with respect to Equation (4). Dickson [**6**] gives a solution, but ours is more detailed in its treatment of the generating formula.

It would be interesting to find other calculus examples that lead to interesting problems in number theory. We looked at a number of other optimization problems in Stewart [**18**] but for each one it was either trivial to determine when the solutions were rational or m-rational, or the solutions were never "nice."

From three-factor polynomials to $a^2 + kb^2 = c^2$

If we take the first and second derivatives of Equation (2), we get

$$\begin{aligned}
\frac{dy}{dx} &= (2m+1)x^{2m} - (A+B)(m+1)x^m + AB,\\
\frac{d^2y}{dx^2} &= 2m(2m+1)x^{2m-1} - m(A+B)x^{m-1}\\
&= x^{m-1}\big(2m(2m+1)x^m - m(A+B)\big).
\end{aligned}$$

It is immediate that the second derivative has zeros when $x = 0$ or when x is the mth root of a rational number. On the other hand, the first derivative is a quadratic in x^m and so its roots will be mth roots of rational numbers only when this quadratic has rational roots.

If we apply the quadratic equation, we see that $\frac{dy}{dx} = 0$ when

$$x^m = \frac{(A+B)(m+1) \pm \sqrt{(m+1)^2A^2 + (2m^2 - 4m - 2)AB + (m+1)^2B^2}}{2(2m+1)}.$$

Therefore, x^m is rational exactly when the discriminant

$$\sqrt{(m+1)^2A^2 + (2m^2 - 4m - 2)AB + (m+1)^2B^2}$$

is rational. If we remove a factor of B^2, then beneath the radical we get

$$(m+1)^2\left(\frac{A}{B}\right)^2 + (2m^2 - 4m - 2)\frac{A}{B} + (m+1)^2.$$

For this quantity to be the square of a rational number there must exist $f,\ g \in \mathbb{N}$, f and g coprime, such that

$$(m+1)^2\left(\frac{A}{B}\right)^2 + (2m^2 - 4m - 2)\frac{A}{B} + (m+1)^2 - \left(\frac{f}{g}\right)^2 = 0.$$

We again apply the quadratic formula to solve for the ratio $\frac{A}{B}$:

$$\frac{A}{B} = \frac{1 + 2m - m^2 \pm \sqrt{(m^2 - 2m - 1)^2 - (m+1)^2\left[(m+1)^2 - \left(\frac{f}{g}\right)^2\right]}}{(m+1)^2}. \tag{5}$$

The right-hand side is rational when there exist $p,\ q \in \mathbb{N}$ coprime such that

$$(m+1)^2\left(\frac{f}{g}\right)^2 - 4m^2(2m+1) = \left(\frac{p}{q}\right)^2;$$

equivalently,

$$p^2g^2 + 4(2m+1)m^2g^2q^2 = q^2f^2(m+1)^2.$$

In other words, we have shown that if the solution to $\frac{dy}{dx} = 0$ is m-rational, then we get a solution to the Diophantine equation

$$a^2 + (2m+1)b^2 = c^2, \quad a,\ b,\ c \in \mathbb{N}. \tag{6}$$

The general Diophantine equation

Though $2m + 1$ is odd, we will consider the Diophantine equation

$$a^2 + kb^2 = c^2, \quad a,\ b,\ c \in \mathbb{N}, \tag{7}$$

for any $k \in \mathbb{N}$. To find all solutions of this equation it will suffice to find all triples with no common factor; thus we define a primitive triple (a, b, c) to be any solution of (7) such that a, b, c are coprime.

Theorem 3. *Given $k \in \mathbb{N}$, let u and v be natural numbers such that* $\gcd(u, v) = 1$ *and $u \geq \frac{v}{\sqrt{k}}$, and let r be the greatest common divisor of $ku^2 - v^2$ and $2uv$. Then*

$$a = \frac{ku^2 - v^2}{r}, \quad b = \frac{2uv}{r}, \quad c = \frac{ku^2 + v^2}{r},$$

give all primitive triples (a, b, c) to Equation (7).

An interesting, hidden feature of this generating formula is that the factor r can only take on a limited number of values. When $k \equiv 1, 2, 3 \pmod 4$, we have the following elegant result.

Theorem 4. *Let $k \equiv 1, 2, 3 \pmod 4$ and let u and v be as in Theorem 3. Then r can only be equal to either $\gcd(v, k)$ or $2\gcd(v, k)$; the exact value depends on the parity of u and v and is given in Table 1.*

TABLE 1: Possible values for r when $k \equiv 1, 2, 3$.

	Value of r		
parity	$k \equiv 1$	$k \equiv 2$	$k \equiv 3$
u odd, v odd	$2\gcd(v, k)$	$\gcd(v, k)$	$2\gcd(v, k)$
u odd, v even	$\gcd(v, k)$	$\gcd(v, k)$	$\gcd(v, k)$
u even, v odd	$\gcd(v, k)$	$\gcd(v, k)$	$\gcd(v, k)$

We illustrate these cases with some examples of primitive triples (a, b, c) for various values of k and the generating triple $\langle u, v, r\rangle$. See Table 2.

TABLE 2: Primitive triples when $k \equiv 1,\ 2,\ 3$.

parity	$k \equiv 1$	$k \equiv 2$	$k \equiv 3$
u odd, v odd	$k = 5$	$k = 2$,	$k = 3$
	$\langle 3, 1, 2\rangle$,	$\langle 5, 1, 1\rangle$,	$\langle 3, 1, 2\rangle$,
	$(22, 3, 23)$	$(49, 10, 51)$	$(13, 3, 14)$
u odd, v even	$k = 5$	$k = 6$	$k = 7$
	$\langle 3, 4, 1\rangle$	$\langle 3, 2, 2\rangle$	$\langle 1, 2, 1\rangle$
	$(29, 24, 61)$	$(25, 6, 29)$	$(3, 4, 11)$
u even, v odd	$k = 9$	$k = 2$	$k = 3$
	$\langle 2, 3, 3\rangle$	$\langle 4, 3, 1\rangle$	$\langle 4, 3, 3\rangle$
	$(9, 4, 15)$	$(23, 24, 41)$	$(13, 8, 19)$

When $k \equiv 0 \pmod 4$, the restrictions on r, u, and v are slightly more complicated. Rewrite $k = 2^{2i}\lambda$, where $\lambda \equiv 1, 2, 3 \pmod 4$ and $i \in \mathbb{N}$. Then finding all primitive solutions to (7) is equivalent to finding all primitive solutions to the equation

$$a^2 + \lambda\beta^2 = c^2, \tag{8}$$

where $\beta = 2^i b$. We can do this by applying Theorem 4; however, to express our solution it is easier to modify the generating formulas.

Theorem 5. *Let $k \equiv 0 \pmod 4$, and fix $\lambda \equiv 1, 2, 3 \pmod 4$ and $i \in \mathbb{N}$ such that $k = 2^{2i}\lambda$. Then*

$$a = \frac{\lambda u^2 - v^2}{\gcd(v, \lambda)}, \quad b = \frac{2uv}{2^i \gcd(v, \lambda)}, \quad c = \frac{\lambda u^2 + v^2}{\gcd(v, \lambda)}$$

give all primitive triples to Equation (7) for k as given and for u and v coprime, $u \geq \frac{v}{\sqrt{\lambda}}$, subject to the restrictions in Table 3.

TABLE 3: Additional restrictions on u and v when $k \equiv 0$. N/A denotes a case which does not occur. The restrictions on u and v may give no extra information if $i = 1, 2$.

	$k \equiv 0$	
parity	$\lambda \equiv 1, 3$	$\lambda \equiv 2$
u, v odd	N/A	i=1 only
u odd, v even	$2^{i-1}\|v$	$2^{i}\|v$
u even, v odd	$2^{i-1}\|u$	$2^{i-1}\|u$

We now prove Theorems 3 through 5. To prove the generating formula in Theorem 3 we start with a given primitive triple and proceed in a fashion similar to that used to find all Pythagorean triples (cf. Courant & Robbins [**5**]).

Proof of Theorem 3. Let (a, b, c) be a primitive triple. Rewrite (7) as $\frac{c+a}{kb} = \frac{b}{c-a}$, and set this equal to $\frac{u}{v}$, where $u, v \in \mathbb{N}$ are coprime. We can then solve for the ratios $\frac{a}{b}$ and $\frac{c}{b}$:

$$\frac{a}{b} = \frac{ku^2 - v^2}{2uv} \quad \text{and} \quad \frac{c}{b} = \frac{ku^2 + v^2}{2uv}.$$

Note that this implies that $u \geq \frac{v}{\sqrt{k}}$.

The left-hand side of each equation is a fraction in lowest terms, and if the right-hand side were also in lowest terms, then we could equate numerators and denominators. However, in general this will not be the case. Let r be the greatest common factor of $ku^2 - v^2$ and $2uv$; then, since both fractions have denominator b on the left-hand side, we must also have that $r = \gcd(2uv, ku^2 + v^2)$. Therefore, we have the relation

$$a = \frac{ku^2 - v^2}{r}, \quad b = \frac{2uv}{r}, \quad c = \frac{ku^2 + v^2}{r}. \tag{9}$$

Conversely, given any choice of u and v with $\gcd(u, v) = 1$ and $u \geq \frac{v}{\sqrt{k}}$, define r as above. Then it is straightforward to show that (a, b, c) is a primitive triple. Therefore, the parameterization (9) completely characterizes the primitive solutions of the Diophantine Equation (7). ■

The proof of Theorem 4 is a rather involved argument that looks at the prime factors of r modulo 4 to determine the possible values of r.

Proof of Theorem 4. We will show that r must always be equal to $\gcd(v, k)$ or $2\gcd(v, k)$ and then consider the relationship between the parity of u and v and the values of r. For brevity, let $(v, k) = \gcd(v, k)$, and define k_0 and v_0 by $k = k_0(v, k)$ and $v = v_0(v, k)$. Then $\gcd(v_0, k_0) = 1$ and we have that

$$r = \gcd\left(k_0(v,k)u^2 - v_0^2(v,k)^2, 2uv_0(v,k), k_0(v,k)u^2 + v_0^2(v,k)^2\right). \tag{10}$$

Hence, $(v, k)|r$ and so we can rewrite r as $r = r_0(v, k)$. Suppose first that $r_0 \neq 2^j$, $j \geq 0$. Then there exists a prime $q \neq 2$ that divides r_0; in particular, a larger power of q must divide r than divides (v, k). Therefore, by (10) we must have that $q|2uv_0$; since $q \neq 2$, $q|u$ or $q|v_0$. Suppose $q|u$; since we also have that $q|k_0u^2 \pm v_0^2(v, k)$, we have that q divides $v_0^2(v, k)$ which is impossible since u and v are coprime. On the other hand, suppose $q|v_0$. Then, since $q|2k_0u^2$, and v_0 and k_0 are relatively prime, $q|u$ which contradicts the fact that u and v are coprime.

Therefore, we must have that $r_0 = 2^j$, $j \geq 0$. We will now show that $j = 0, 1$. Suppose to the contrary that $j \geq 2$. Then arguing as before we must have that $r_0|2uv_0$, so $2|u$ or $2|v_0$. If $2|u$, then we get $2|v$, a contradiction. If $2|v_0$ we get $2|k_0u^2$, and since $\gcd(v_0, k_0) = 1$, $2|u$ which is again a contradiction. Therefore, $r_0 = 1,\ 2$.

We now show that the value of r_0 depends on the parity of u and v. We consider the three cases $k \equiv 1,\ 2,\ 3 \pmod 4$ in turn. For brevity, all equivalences should be read as modulo 4. First suppose that $k \equiv 1$. If u and v are both odd, then $ku^2 - v^2 \equiv 0$, $ku^2 + v^2 \equiv 2$, and $2uv \equiv 2$. Each is divisible by 2, but (v, k) is odd. Hence, $r = 2(v, k)$. If u is odd and v is even, then, $ku^2 - v^2 \equiv 1$, $ku^2 + v^2 \equiv 1$, and $2uv \equiv 0$. There is no common factor of 2, so $r = (v, k)$. Finally, if u is even and v odd, then $ku^2 - v^2 \equiv 3$, $ku^2 + v^2 \equiv 1$, and $2uv \equiv 0$. There is again no common factor of 2, and so $r = (v, k)$.

Now suppose $k \equiv 2$. If u and v are both odd, then, $ku^2 - v^2 \equiv 1$, $ku^2 + v^2 \equiv 3$, and $2uv \equiv 2$, and so $r = (v, k)$. If u is odd and v is even, then $ku^2 - v^2 \equiv 2$, $ku^2 + v^2 \equiv 2$, and $2uv \equiv 0$. There is a common factor of 2, but (v, k) is even, and since not all of the terms are divisible by 4, we must have $r = (v, k)$. If u is even and v is odd, then $ku^2 - v^2 \equiv 3$, $ku^2 + v^2 \equiv 1$, and $2uv \equiv 0$, so $r = (v, k)$.

Finally, suppose $k \equiv 3$. If u and v are both odd, then, $ku^2 - v^2 \equiv 2$, $ku^2 + v^2 \equiv 0$, and $2uv \equiv 2$. There is a common factor of 2 and (v, k) is odd, so we have $r = 2(v, k)$. If u is odd and v is even, then $ku^2 - v^2 \equiv 3$, $ku^2 + v^2 \equiv 3$, and $2uv \equiv 0$, so $r = (v, k)$. And if u is even and v is odd, then, $ku^2 - v^2 \equiv 3$, $ku^2 + v^2 \equiv 1$, and $2uv \equiv 0$, so again, $r = (v, k)$. ■

Finally, we prove Theorem 5, which covers the case in which $k \equiv 0 \pmod 4$. The heart of the proof is determining how many powers of 2 can be present in u, v, and r when $k = 2^{2i}\lambda$.

Proof of Theorem 5. Let k, λ, and i be as given. Then by Theorems 3 and 4, all the primitive triples a, β, c for Equation (8) are given by

$$a = \frac{\lambda u^2 - v^2}{r}, \quad \beta = \frac{2uv}{r}, \quad c = \frac{\lambda u^2 + v^2}{r},$$

where u and v are coprime, $u \geq \frac{v}{\sqrt{\lambda}}$, and r is from Table 1. To get the desired solutions to Equation (7) we need to account for the additional restrictions imposed by the fact that $2^i b = \beta$.

We first consider the case $\lambda \equiv 1, 3 \pmod 4$. Let u and v be odd. Then

$$2^i b = \beta = \frac{2uv}{2\gcd(v, \lambda)} = \frac{uv}{\gcd(v, \lambda)}.$$

Hence, $2^i|uv$. However, u and v are both odd, so this is impossible. Therefore, when $\lambda \equiv 1, 3$ we must have that $r = \gcd(v, \lambda)$. Now, let u be odd and v even. Then

$$2^i b = \beta = \frac{2uv}{\gcd(v, \lambda)},$$

and so $2^i|2uv$ which gives $2^{i-1}|v$. (When $i = 1,\ 2$ we get no additional information about v.) Finally, let u be even and v odd. Then, $2^i|2uv$, so $2^{i-1}|u$. (Again, we get no additional information when $i = 1,\ 2$.)

Next we consider the case $\lambda \equiv 2 \pmod 4$. By Theorem 4, $r = \gcd(v, \lambda)$. Let u and v odd; then

$$2^i b = \beta = \frac{2uv}{\gcd(v, \lambda)}.$$

Since $\gcd(v, \lambda)$ is odd, $2^{i-1} | uv$, but this is only possible when $i = 1$. Now let u be odd and v even. Then

$$2^i b = \beta = \frac{2uv}{\gcd(v, \lambda)} = \frac{uv}{\gcd(v/2, \lambda/2)}.$$

Hence, $2^i | uv$, so $2^i | v$. Finally, let u be even and v odd. Then

$$2^i b = \beta = \frac{2uv}{\gcd(v, \lambda)},$$

so $2^{i-1} | u$. If we combine the results from both cases, we get the desired conclusion. ■

Solving the original problem

Given Theorems 3 and 4, we can now reverse the steps of our analysis and find all three-factor polynomials whose first derivatives have m-rational roots, thus proving Theorem 2. Since $k = 2m + 1$ we only need to consider the cases $k \equiv 1,\ 3 \pmod 4$ from Theorem 4. We will then take $m = 1$ and prove Theorem 1 and so characterize the rational solutions of the box problem.

Proof of Theorem 2. Recall that in our original reduction to the Diophantine Equation (6) we set $b = 2gmq$ and $c = (m + 1)fq$. If we solve for the ratio $\frac{f}{g}$ and combine this with our generating formulas (9) from Theorem 3 we get

$$\frac{f}{g} = \left(\frac{c}{b}\right)\left(\frac{2m}{m+1}\right) = \left(\frac{(2m+1)u^2 + v^2}{2uv}\right)\left(\frac{2m}{m+1}\right).$$

If we substitute this into (5) and simplify, then we have Equation (3):

$$\frac{A}{B} = \frac{1 + 2m - m^2}{(m+1)^2} \pm \frac{2m}{(m+1)^2}\sqrt{\left(\frac{(2m+1)u^2 + v^2}{2uv}\right)^2 - (2m+1)}.$$

Note that since we are concerned with ratios, the factor r disappears and our formula only depends on m and our choices of u and v. Theorem 3 gives us the restrictions $\gcd(u, v) = 1$ and $u \geq \frac{v}{\sqrt{2m+1}}$. This completes the proof. ■

The following refinement of Theorem 2 is useful to prove Theorem 1.

Corollary 6. *If we restrict $A, B > 0$ or $A, B < 0$, then it suffices to choose $u \geq v$,* $\gcd(u, v) = 1$, *and the negative branch of Equation* (3) *is impossible.*

Proof. Since A and B must have the same sign, without loss of generality we may assume $\frac{A}{B} \geq 1$. Given this, if we denote the expression under the radical by R, then we must have that

$$1 + 2m - m^2 \pm 2m\sqrt{R} \geq (m+1)^2,$$

which implies that $\pm\sqrt{R} \geq m$, so the negative branch is impossible. Moreover, $R \geq m^2$ which implies

$$(2m+1)u^2 + v^2 \geq 2(m+1)uv.$$

Since $u \geq \frac{v}{\sqrt{2m+1}}$, if we fix a, $0 < a \leq \sqrt{2m+1}$ such that $u = \frac{v}{a}$, then we may rewrite the above inequality as the quadratic expression

$$a^2 - 2(m+1)a + (2m+1) \geq 0.$$

This holds provided that a is not between the two roots of the quadratic:

$$a = \frac{2(m+1) \pm \sqrt{4(m+1)^2 - 4(2m+1)}}{2} = 1,\ 2m+1.$$

Since $a \leq \sqrt{2m+1}$, we must have $a \leq 1$. In other words, to find ratios $\frac{A}{B} \geq 1$, we should take $u \geq v$. ■

Theorem 1 now emerges as a corollary to the work done above by setting $m = 1$.

Proof of Theorem 1. Since we assume that $A \geq B > 0$, by Corollary 6, we need only take $u \geq v$ and we eliminate the negative branch of Equation (3). Thus, if we set $m = 1$, we get that all solutions to (1) are given by

$$\frac{A}{B} = \frac{1}{2} + \frac{3u^2 - v^2}{4uv}.$$

The change of variables that yielded (1) does not change the ratio of solutions, so this equation gives all the solutions to the original box problem. ■

A brief history of the problem

As we noted at the beginning, the box problem itself is now a staple of calculus textbooks. Besides Stewart [**18**], a random check of current books showed that it is also in Larson and Edwards [**12**], Rogawski [**17**], and Varberg, Purcell and Rigdon [**21**]. Going back in time, it is in the classic calculus book by Thomas [**19**]. Going further back, it can be found in books from 1917 [**14**], 1909 [**13**], and 1875 [**16**]. The earliest appearance we could find was in the 1852 text *A Treatise on the Differential Calculus* by Isaac Todhunter [**20**]. However, we suspect the problem was considered even earlier in some form.

The problem of finding dimensions for the rectangle that yields rational solutions has also been considered by several different authors: see Dundas [**9**], Coll *et al.* [**4**] (who give the same solution as Dickson [**6**]); Graham and Roberts [**11**], Duemmel [**8**] (who confronted the same problem writing an exam as did the second author); Buddenhagen *et al.* [**2**]. (Hereafter, we adopt the notation of our sources for ease of reference). The last three papers all begin their analysis in a way similar to us but focus on the Diophantine equation $a^2 + b^2 - ab = c^2$, though Graham and Roberts transform this equation to the same form as ours: $n^2 + 3m^2 = c^2$. All of these authors generalized their results in a variety of ways, but none considered the general Diophantine Equation (4). Dundas's exhaustive treatment of differently shaped boxes was the initial inspiration for our generalization to three-factor polynomials.

Three-factor polynomials do not seem to have appeared in the literature, at least in this form. However, a much deeper generalization of the original question is the problem of finding polynomials with integer coefficients such that they and all their derivatives have rational roots. For a survey of this problem with extensive references, see Buchholz and MacDougall [**1**].

Equation (4) has a long and interesting history. One of the earliest treatments of this specific equation is found in the work of the Japanese mathematician Matsunaga from the first half of the eighteenth century [**15**] (pp. 229–232). He dealt with the equation $rx^2 + y^2 = z^2$, and gave the infinite family of solutions $x = 2mn$, $y = rm^2 - n^2$, $z = rm^2 + n^2$, where n is "even or odd according as r is odd or even," and not divisible by r. This is similar to our solutions, but he apparently did not analyze the subtleties of the different cases. The historical context for Matsunaga's work on the problem is intriguing: he was part of the *Rangaku* ("Dutch Study") movement during the Tokugawa shogunate: a group of Japanese scholars who kept up with Western science through contact with the Dutch at the trading post of Dejima, the only foreigners allowed in Japan during this isolationist period [**10**].

In Europe, Equation (4) was probably studied by the early 18th century. More general equations were treated, at least in part, by Lagrange ($Ar^2 = p^2 - Bq^2$), Euler ($\alpha f^2 + \beta g^2 = \gamma h^2$), Minding ($x^2 = Ay^2 + Bz^2$), and Dirichlet ($ax^2 + by^2 + cz^2 = 0$): see Dickson, *History of the Theory of Numbers, Vol. 2* (pp. 420–423) [**7**]. Solutions of this exact equation can be found in Carmichael, *Diophantine Analysis* [**3**] and Dickson, *Introduction to the Theory of Numbers* [**6**]. Carmichael, through other methods, shows that

$$x = mn^2 + Dn^2, \quad y = 2mn, \quad z = m^2 - Dn^2$$

is *a* set of solutions, but then proceeds to prove it is the general set of solutions in a rather roundabout way: he first solves $x^2 - Dy^2 = 1$ for rational solutions, then introduces a number of auxiliary variables and uses them to find the special cases when these solutions are integers. Then he arrives at a method of generating solutions to $x^2 - Dy^2 = \sigma^2$ for a chosen σ, and explains that "In order to apply the theorem in a particular case it is necessary first to find, by inspection or otherwise, the least positive integral solution." Dickson gives a solution similar to ours. Neither author, however, gives the analysis of the generating terms that we give in Theorems 4 and 5.

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Summary. We consider a well known calculus question, and show that the solution of this problem is equivalent to finding integer solutions to a Diophantine equation. We generalize the calculus question, which in turn leads to a more general Diophantine equation. We give solutions to all of these and describe some of the historical background.

DAVID CRUZ-URIBE, OFS (MR Author ID: 329597 is professor and chair of mathematics at the University of Alabama in Tuscaloosa, AL. Prior to this, he spent nineteen years on the faculty at Trinity College, Hartford, CT. His research is in harmonic analysis and partial differential equations; this foray into number theory was fascinating but unintended.

GREGORY CONVERTITO (MR Author ID: 1146505 is a senior at Trinity College in Hartford, Connecticut. He studies mathematics and philosophy, and enjoys sailing and sitting on the beach, reading.

Ask Siri . . .

A: What is 4 divided by 0?
S: The answer is definitely . . . undefined.

$$4 \div 0 = \text{undefined}$$

Submitted by Anneliese Jones, Ann Arbor, MI

Proof Without Words: Sums of Reciprocals of Binomial Coefficients

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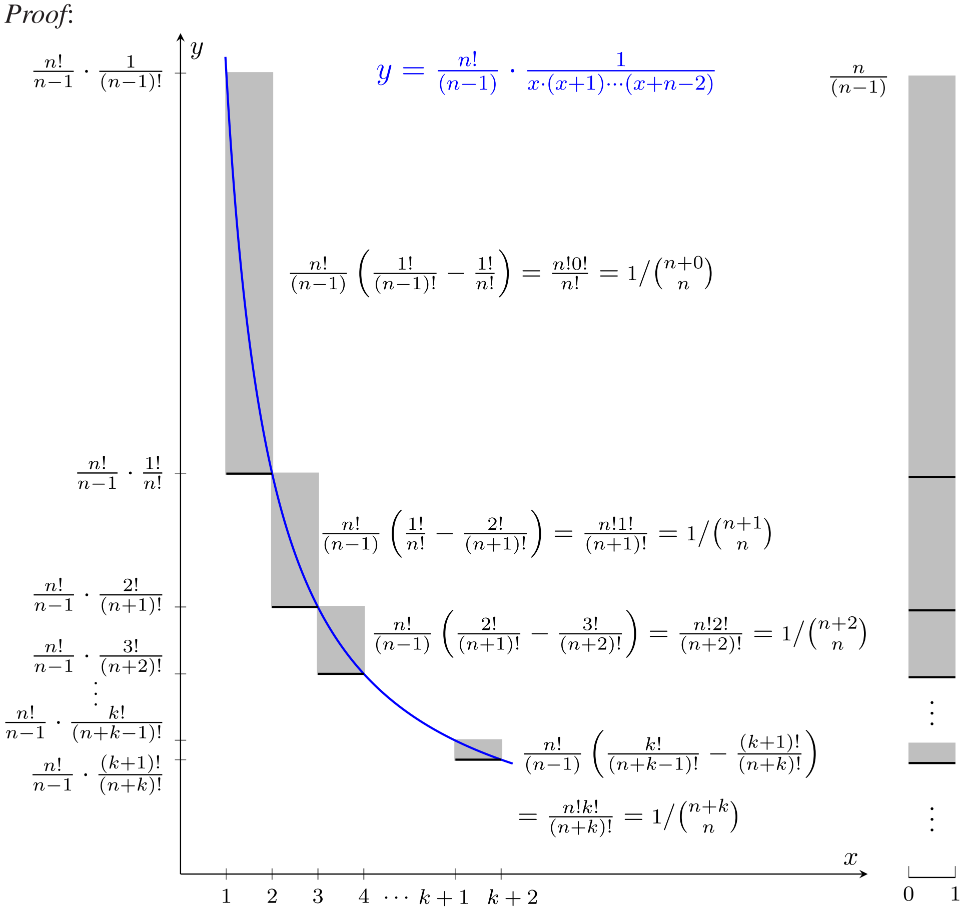
In what follows, n is an integer greater than 2 and $\binom{n+k}{n}$ is the binomial coefficient $\dfrac{(n+k)!}{n!\cdot k!}$.

Theorem.

$$\sum_{k=0}^{\infty} 1\Big/\binom{n+k}{n} = 1\Big/\binom{n+0}{n} + 1\Big/\binom{n+1}{n} + 1\Big/\binom{n+2}{n} + \cdots = \frac{n}{n-1}.$$

Proof:



■

Math. Mag. **89** (2016) 212–213. doi:10.4169/math.mag.89.3.212. © Mathematical Association of America
MSC: Primary 05A10.

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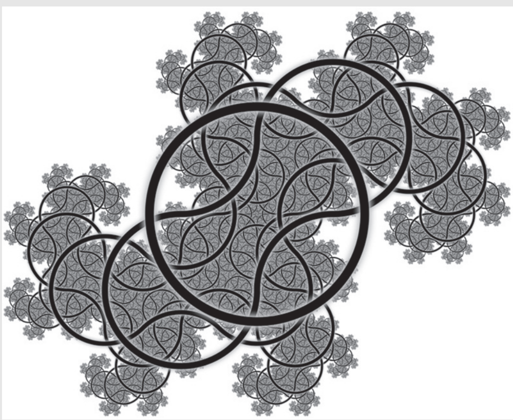
2. R. B. Nelsen, Proof without Words: Sum of Reciprocals of Triangular Numbers. *Math. Mag.*, **64** no. 3 1991, 167.

Summary. We provide a visual computation of the sum of the series obtained by adding the reciprocals of entries from column n from Pascal's triangle.

TOM EDGAR (821633) is an associate professor of mathematics at Pacific Lutheran University in Tacoma, Washington. He enjoys investigating the combinatorics and number theory associated with various triangular arrays.

Artist Spotlight
Robert Fathauer



Twenty Generations of Dragons, Robert Fathauer; 16 in. × 20 in. limited edition of 50 digital prints, 2013. ”Twenty Generations of Dragons” shows 20 generations of an iterative modification of the ”dragon” curve. This particular version of the dragon curve has smooth, closed curves, with the first generation being a circle.

See interview on page 220.

Ask Siri . . .

A: What is 5 divided by 0?
S: The same as 6 divided by 0. Undefined.
A: What is 6 divided by 0?
S: The answer is somewhere between infinity, negative infinity, and undefined.

$$6 \div 0 = \text{undefined}.$$

Submitted by Anneliese Jones, Ann Arbor, MI

A Geometric Proof of a Morrie-Type Formula

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Morrie's law is the name of the formula $\cos(20^\circ)\cos(40^\circ)\cos(80^\circ) = 1/8$. As mentioned in [**1, 6**], it is the special case $k = 3$ and $x = \pi/9$ of the identity

$$2^k \prod_{j=0}^{k-1} \cos(2^j x) = \frac{\sin(2^k x)}{\sin x}.$$

In this short contribution, we are concerned with the case $k = 3$ and $x = \pi/7$ of the above formula, which specializes to

$$\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right) = \frac{1}{8}, \tag{1}$$

where we have explicitly used that $\cos(4\pi/7) = -\cos(3\pi/7)$.

While Morrie's law and formula (1) are merely trigonometric curiosities, they share the remarkable fact that none of the six trigonometric factors on their left-hand sides are expressible in terms of ordinary arithmetical operations and finite root extractions on real rational numbers. This is a consequence of an old and beautiful result of Gauss and Wantzel [**2, 3, 5**] that states that for a positive integer n, the values of $\cos(\pi/n)$ can be expressed in terms of ordinary arithmetical operations and finite root extractions on real rational numbers if and only if

$$n = 2^k p_1 p_2 \cdots p_s, \tag{2}$$

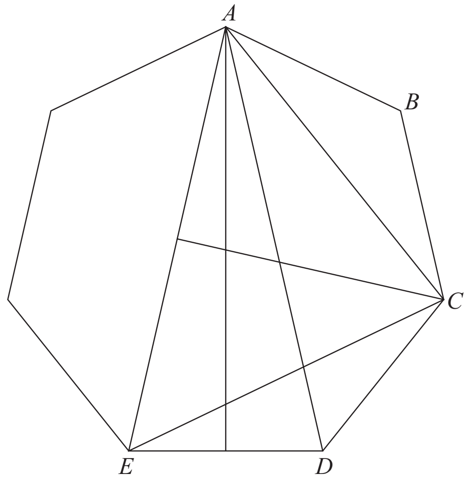
where k is a nonnegative integer, and $p_1, p_2, \ldots, p_s$ is a finite collection (maybe empty) of distinct Fermat primes. Recall that a Fermat prime is a prime number of the form $F_k = 2^{2^k} + 1$ for some nonnegative integer k. The only known Fermat primes are the first five Fermat numbers $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$. Since 7 and 9 are the first positive integers that are not expressible in the form (2), then it is not possible to get a closed-form expression for $\cos(\pi/7)$, $\cos(\pi/9)$, and the other four factors in terms of ordinary arithmetical operations and finite root extractions on real rational numbers.

As we did in [**4**], in order to give a geometric proof of (1), we consider the regular heptagon with unit edge length in which some of its diagonals and angle bisectors are drawn, as shown in the figure below.

First, since $\angle B = 5\pi/7$, then $\angle BAC = \angle BCA = \pi/7$, and thus, $|AC| = 2\cos(\pi/7)$. Next, $\angle CAE = 2\pi/7$ because $\angle ACE = 5\pi/7 - 2(\pi/7) = 3\pi/7$, and hence, $|AE| = 2|AC|\cos(2\pi/7)$. Finally, $\angle AED = \angle AEC + \angle CED = 3\pi/7$ so that $|ED| = 2|AE|\cos(3\pi/7)$.

A backward substitution gives us $|ED| = 2^3\cos(\pi/7)\cos(2\pi/7)\cos(3\pi/7)$. But $|ED| = 1$, which completes the proof.

Math. Mag. **89** (2016) 214–215. doi:10.4169/math.mag.89.3.214.
MSC: Primary 51A00



Exercise. Find a geometric proof of the formula

$$\cos(\pi/5)\cos(2\pi/5) = 1/4.$$

Acknowledgment The authors thank the referee for valuable suggestions that have improved the presentation of this note.

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5. M. L. Wantzel, Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas, *J. Math. Pures Appl.* **2** (1837) 366–372.
6. G. J. Tee, Further Generalizations of a Curiosity that Feynman Remembered All His Life, *Math. Mag.* **72** (1999) 44.

Summary. We use a regular heptagon to give a geometric proof of a specific instance of a Morrie-type formula.

SAMUEL G. MORENO (MR Author ID: 721728) graduated from Universidad Autónoma de Madrid (Spain), received an M.S. from Universidad Nacional de San Luis (Argentina) under the direction of Felipe Zó, and did his doctoral studies at Universidad de Jaén (Southern Spain), where he is currently associate professor of mathematics. He enjoys spending time with his family, practicing all kind of sports, and sun and beach when time permits.

ESTHER M. GARCÍA–CABALLERO (MR Author ID: 661661) received her Ph. D. in Mathematics from Universidad de Granada (Spain) and currently serves as vice dean of quality and attention to students with disabilities at E.P.S.L. (Universidad de Jaén). She loves her family, mathematics, and her kitchen for cooking bakes (though, depending on the day, not necessarily in that order).

Proof Without Words: Viviani for Congruent Cevians

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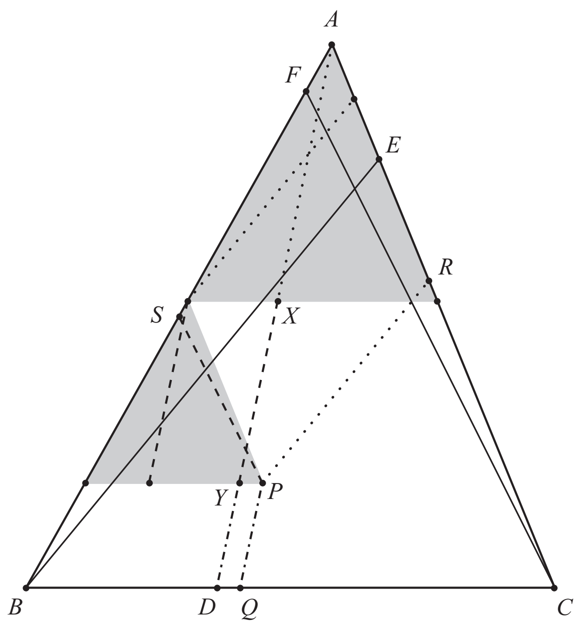
The following results generalize Viviani's theorem about the sum of the distances from the sides of an equilateral triangle.

If point P lies inside a triangle ABC with congruent cevians $AD = BE = CF$, the distances between P and the sides measured parallelly to the cevians sum up to the cevian length:

$$PQ + PR + PS = AD.$$



A few words about the picture prove a well-known more general formula valid for *any* cevians:

$$\frac{PQ}{AD} + \frac{PR}{BE} + \frac{PS}{CF} = 1$$

follows at once from $PR/BE = AX/AD$ and $PS/CF = XY/AD$ or from the area ratios

$$\frac{PQ}{AD} = \frac{[PBC]}{[ABC]}, \quad \frac{PR}{BE} = \frac{[PCA]}{[BCA]}, \quad \frac{PS}{CF} = \frac{[PAB]}{[CAB]}.$$

Math. Mag. **89** (2016) 216–217. doi:10.4169/math.mag.89.3.216.
MSC: Primary 51M04.

Summary. We prove without words that the distances from the sides of a triangle measured parallelly to three congruent cevians sum up to the cevian length. This generalizes Viviani's theorem about the sum of the distances from the sides of an equilateral triangle.

GRÉGOIRE NICOLLIER (MR Author ID: 989132) is a mountain guide and teaches mathematics in French and in German to engineering students at the University of Applied Sciences of Western Switzerland in Sion (Valais). He studied mathematics at ETH (Ph.D. in homological algebra in 1984). His research interests focus on polygons, in particular on discrete dynamical systems related to triangles.

Artist Spotlight
Robert Fathauer

Radial Development, Robert Fathauer; 21 in. × 21 in.× 11 in. ceramic, 2014. This sculpture, which was partly inspired by brain coral, is based on the first three generations of a fractal curve that develops radially outward. The starting point is a simple saddle, and the final form has an envelope that is roughly hemispherical. The space curves were created by fitting a series of planar fractal curves to the surface of an octahedron.

See interview on page 220.

ACROSS

1. Professional mathematician and amateur actor Billingsley to his friends
4. Latch (onto)
8. * Colin from Williams College who will give the MAA Chan Stanek Lecture for Students at MathFest 2016
13. What a *Wheel of Fortune* contestant may buy
14. Sans effort: with ___
15. Genus of tapeworms that attack livestock
16. * Ohio city that will host MathFest 2016
18. Lay flat again, like an origami design
19. Genesis garden
20. * Judy from Kenyon College who will give an MAA Invited Address about mathematics and digital art at MathFest 2016
22. 1984 film starring Patrick Swayze that was remade in 2012 starring Chris Hemsworth
26. Computer operator
27. Soft drink brand named for a mountain in northern California
29. LaTeX command that produces $\top$
32. * Crannell from Franklin & Marshall College who will give the MAA James R.C. Leitzel Lecture about inquiry-based learning at MathFest 2016
35. Ron Howard role as a boy
37. * One of the sponsors of MathFest: Abbr.
38. Estimate the magnitude of
41. Org. that shares many members with 37-Across
42. Sloe gin___
44. * With 43-Down, one of the topics of 8-Across' lecture
46. 2016 Olympics site
47. Spanish dish
49. [smooch]
51. * Hendrik from Universiteit Leiden who will give the three-part Earle Raymond Hedrick Lecture Series at MathFest 2016 about elliptic curves, Nullstellensatz, and number theory
56. * Laba from University of British Columbia who will give the AWM-MAA Etta Z. Falconer Lecture at MathFest 2016 about harmonic analysis
60. Bulls or Bears, e.g.
61. Not requiring much skill
62. * Arthur from Harvey Mudd College who will give an MAA Invited Address about "Magical Mathematics"
66. Main roadway
67. ___ facto
68. Greek letter used for wave functions
69. Not like a rolling stone?
70. ". . . there exists a delta such ___ . . . "
71. WWW technique to boost one's Google presence

DOWN

1. Indiana b-baller
2. + end
3. Covered a chessboard with dominoes, as in classic math problems
4. Jewel
5. Computer program: MAT___
6. * Host sch. of MathFest 2016
7. Triangulation of the domain, in finite element analysis of a boundary-value problem
8. TV network of *Duck Dynasty*, *Intervention*, and *Storage Wars*
9. Prefer the opinion of
10. Auth. unknown
11. 1,760 yards
12. Neighborhood of Baghdad, Iraq: ___ City
15. Oral Roberts University site in Oklahoma
17. State sch. in Grand Forks
21. Away from the office
23. Soundless communication: Abbr.
24. Political party in the United Kingdom (and formerly in America)
25. Discovery grp.
28. Marinara, e.g.
30. Birthstone
31. * One of the sponsors of MathFest: ___ Epsilon
32. Recreational equip. company
33. Hair removal brand
34. Hitler, e.g.
36. Sister of -trix
39. Chutzpah
40. Chantal Akerman (1950-2015) film: *Je Tu Il* ___
43. * With 44-Across, one of the topics of 8-Across' lecture
45. Fire
47. Adam from ABC's *Happy Endings* and FOX's *The Mindy Project*
48. Minor league for the Bruins and Penguins: Abbr.
50. Pooped
52. Amtrak stop: Abbr.
53. Measurements in deg. K
54. Up the ante
55. Prenatal test, for short
56. Muslim leader
57. Aleph ___: cardinality of the integers
58. Aardvark's fare
59. 0 or 1
63. Purple cow that's a mascot for 8-Across' college
64. Biggest employer of mathematicians in the USA
65. Write (down)

Math. Mag. **89** (2016) 218–219. doi:10.4169/math.mag.89.3.218.

MathFest 2016

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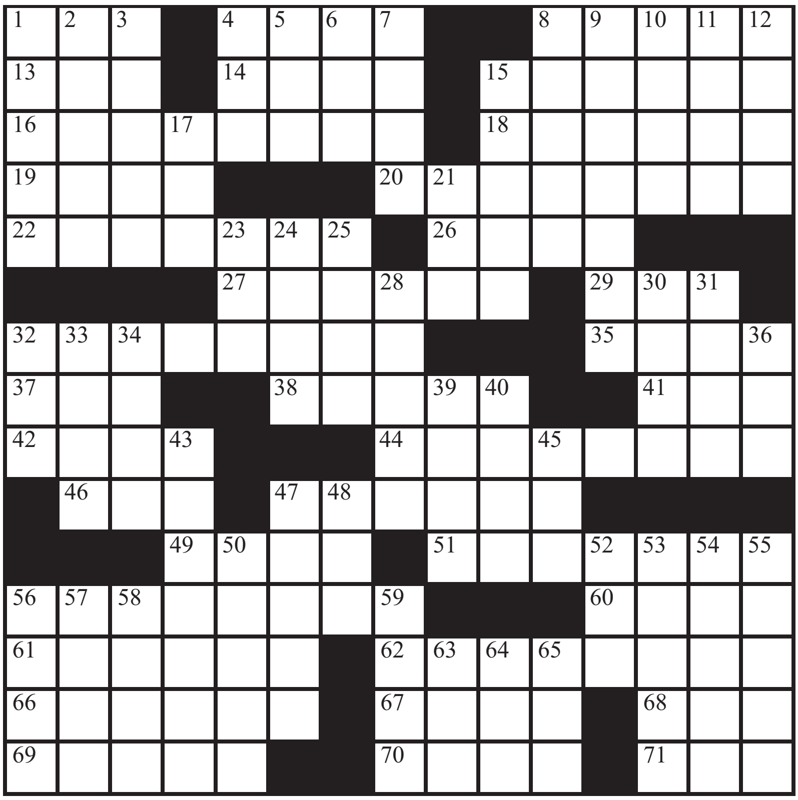

Clues start at left, on page 218. The Solution is on page 188.

Extra copies of the puzzle can be found at the MAGAZINE's website, www.maa.org/mathmag/supplements.

Crossword Puzzle Creators

If you are interested in submitting a mathematically themed crossword puzzle for possible inclusion in MATHEMATICS MAGAZINE, please contact the editor at mathmag@maa.org.

Robert Fathauer: Polymath Purveyor*

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Robert Fathauer is an artist, author, and is the owner and operator of Tessellations. He has taken the lead role in coordinating annual exhibitions of mathematical art at both the Joint Mathematics Meetings and the Bridges Conference, starting in 2004. We met with Robert following the Bridges conference at Mathfest 2015 in Washington, D.C. to discuss his background and art. A portion of this interview appears below. Accompanying artwork appears on the following pages: 176, 178, 195, 213, and 217.

Figure 1 *Four-Fold Development*, Robert Fathauer; 14 in, 2014; ceramic. This sculpture is based on the first five generations of a fractal curve. The starting point is a circle, and the first iteration produces a four-armed form.

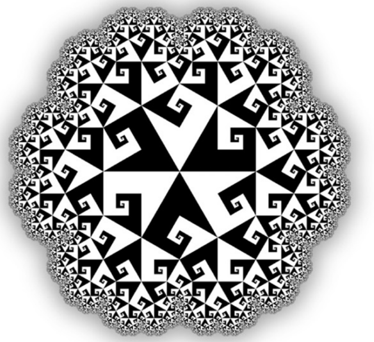
Q: *What is your main occupation, Robert?*

RF: Running Tessellations, a company that I own that produces math-related products, which we sell wholesale to K–12 catalogues and places like MoMath, direct to teachers, and also the website where I retail other people's products. I have been running Tessellations for about 20 years. Originally, I had puzzles, later on added posters, some books and more recently, dice. Over the last two years, I've partnered with Henry Segerman to produce several new unique polyhedral dice, including a die with 120 faces based on the disdyakis triacontahedron.

Q: *How long have you been creating art?*

RF: When I was a kid I took private art lessons starting at age 8 or so, starting with pastels and then I painted oils and acrylics through high school. Not so much in college, and I picked it up some again in graduate school; I started doing some more painting. It wasn't until I graduated with my Ph.D., starting to design my own tessellations, that I consciously started doing math-related art.

Math. Mag. **89** (2016) 220–222. doi:10.4169/math.mag.89.3.220.
MSC: Primary 01A70; Amy L. Reimann (MR Author ID: 1118776) and David A. Reimann (MR Author ID: 912704)



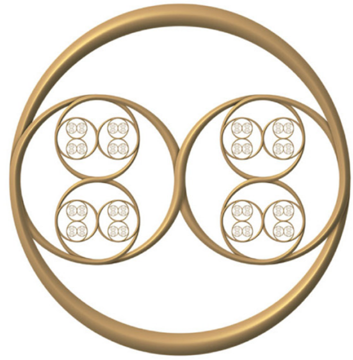


Figure 2 *Fractal Tessellation of Spirals* (left), Robert Fathauer; 4.5 in × 12 in, 2011; digital print. This work is based on a fractal tessellation of kite-shaped tiles discovered by Fathauer. All of the spirals are similar in the Euclidean plane.
Infinity (right), Robert Fathauer; 13 in × 13 in, 2006; limited digital print. Starting with the trefoil knot, iterations incorporate two smaller copies within loops in the starting knot such that the resulting more complex knot remains unicursal.

Q: *So let's talk a little bit about your educational background.*

RF: As an undergraduate I majored in physics—I double majored in math, but the emphasis was on the physics, at the University of Denver. And then in graduate school, I started in applied physics and transferred to electrical engineering, so my Ph.D. is in electrical engineering from Cornell University.

Q: *What mediums do you like to work in?*

RF: I enjoy creating digital art. I like working with a program called Freehand, which is similar to Illustrator, and also Photoshop to create geometric forms and artworks based on them, and I like ceramics too. The last two years I've been taking ceramics classes at an art center in Mesa. It's real different. I think it's good too—it forces me to get away from the computer and use my hands. I've also got other artists in the studio, and it's kind of nice to be around other artists than in a room by myself in front of a computer.

Q: *What are your inspirations for your art?*

RF: My original inspiration for getting into math-related art was Escher's work, and I guess it's still the biggest inspiration. I think nature, interesting natural forms, usually related to math, symmetry, primarily tiling, and more recently, fractals, knots, and polyhedra are inspirations. I also like Van Gogh and Andrè Derain—he was a member of the Fauve movement—bright colored landscapes, that kind of thing. I would say architecture is another inspiration. I like Frank Lloyd Wright, and I like Greene and Greene, the Craftsman architects, and Charles Mackintosh. And Japanese design, too.

Q: *If we wanted to visualize what your studio space looks like, tell us about where you create.*

RF: I don't have a dedicated studio. My digitals I do in my office. I've got a laptop and a big second monitor above that. I now have a 3D printer I bought recently that's also in that room. In my garage, we have kind of a side room that's a work room, which I have set up to photograph my work, like ceramics, but also I do some woodwork and stuff out there. My office has some of my own art. And I also have a kind of math curio case in my office, so it's a little bit workspace, a little bit math art space. I also have

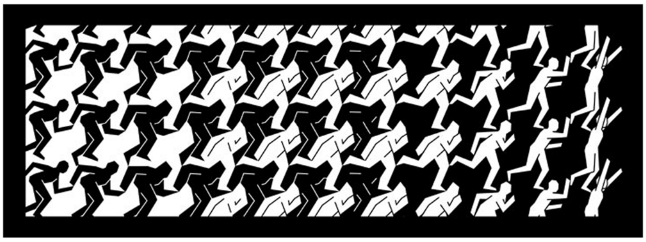


Figure 3 *Marathon*, Robert Fathauer; 4.5 in × 12 in, 2004; limited digital print. Originally commissioned for the 2004 Chicago Marathon, this print depicts a runner progressing, via metamorphosis and figure/ground reversal, to the finish line of a race.

a row of African masks and some carved animals. Then I've got a case that has some mineral specimens, seashells, and kaleidoscopes.

Q: *Do you collect anybody's specific art? Do you collect art of other people?*

RF: I have one original Escher I bought a long time ago. I also have a few pieces by the Hungarian artist István Orosz; he's another artist I watch. He often uses optical illusions in his art. I met him at the Escher Centennial in '98, and used his art for a jigsaw puzzle and a poster, and I have a few of his prints.

Q: *What are other places that people can see your art?*

RF: Mostly online. My own art site is www.robertfathauer.com. I have some stuff on DeviantArt, all the Bridges exhibits are online. I have some murals in Rock Valley College in Rockport, Illinois, in a science building, built about five years ago. The architect of the project found me online and wanted to do some study areas in one wide hallway with chairs and tables, and three or four of those with walls covered by murals made up from my fractal trees. She did multiple images of a fractal tree for each mural in different shades. That's the one public installation I have at this point in time.

Q: *You've also published a number of books, right?*

RF: I've written several books, mostly on tessellations and designing them and tessellated polyhedra and *Tessellations Around the World* just came out; it's my newest one. And then the *Fractal Trees* book. T-shirts, I've been doing just recently also. Mainly sell those retail, I don't really wholesale them—well, except I have sold some to MoMath. I sold a lot of them for Pi Day.

Q: *Do you make art to sell and do you take commission type work?*

RF: Yes, I do sell my art. I don't sell much art because I don't do much to market it other than have it online or show it at a conference. I do commission work, too. I have a morph of a runner from starting to finish, maybe you've seen it, it's called Marathon (see Figure 3). That was a commission for the Chicago Marathon, for a T-shirt design. And I do license work if people are interested.

Q: *What else would you like us to know? What would you like people who read Mathematics Magazine to know?*

RF: Watching the tremendous growth over the past decade in the art exhibitions at the Joint Mathematics Meetings and the Bridges Conference has been very rewarding!

PROBLEMS

Proposals

To be considered for publication, solutions should be received by November 1, 2016.

1996. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Compute

$$\lim_{x\to 0} \frac{1}{x}\int_0^x \left|\cos\frac{1}{t}\right| dt.$$

1997. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

$$\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^2 dx.$$

1998. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.*

Let $\mathbb{N}$ be the set of natural numbers. We call a collection $\mathcal{C}$ of subsets of $\mathbb{N}$ *plenary* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $f[X] = \mathbb{N}$ for all $X \in \mathcal{C}$, where $f[X] = \{f(x) : x \in X\}$ is the set of images of elements of X under f.

Math. Mag. **89** (2016) 223–230. doi:10.4169/math.mag.89.3.223.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Effective immediately, authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We hope that this online system will help streamline our editorial team's workflow while still proving accessible and convenient to longtime readers and contributors. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

(a) Prove that any countable collection of infinite subsets of $\mathbb{N}$ is plenary.

(b) Prove that the collection of all infinite subsets of $\mathbb{N}$ is not plenary.

(c) Are there any uncountable plenary collections?

1999. *Proposed by Mihàly Bencze, Brasov, Romania.*

For any real number $a > 1$, evaluate

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{a^m(na^m + ma^n)}.$$

2000. *Proposed by Michel Bataille, Rouen, France.*

Let $\triangle ABC$ have a right angle at A. Let M be the midpoint of AB, let D lie on side $\overline{BC}$ so $BD = BA$, and let P lie on the circumcircle of $\triangle ADC$ so that $\angle APB = 90^\circ$. Let U lie on line $\overleftrightarrow{AP}$ so that $\overline{BU}$ is perpendicular to $\overline{MP}$, and let V lie on $\overleftrightarrow{DP}$ so that $\overline{BV}$ is parallel to $\overline{MP}$.

Prove that $PU/PV = BU/BV$ and the line $\overleftrightarrow{CP}$ bisects $\overline{UV}$.

Quickies

1061. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.*

If $x^2 = 1 + zx^3$ and $y^2 = 1 - zy^3$ with $x, y > 0$, prove that $xy = 1 + z^2x^3y^3$.

1062. *Proposed by Julien Sorel, Piatra Neamt, Romania.*

Prove that

$$\int_{\sqrt{3}/3}^{\sqrt{3}}\frac{dt}{(1+t^2)^2} = \frac{\pi}{12}$$

without finding the antiderivative of the integrand.

Solutions

An application of Brahmagupta's identity **April 2015**

1966. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let n be a square-free natural number. Let S be an infinite set of integer quadruples (a, b, c, d) such that the sets $\{ad - bc : (a, b, c, d) \in S\}$ and $\{ac - nbd : (a, b, c, d) \in S\}$ are bounded. Prove that the set $\{a^2 - nb^2 : (a, b, c, d) \in S\}$ is bounded.

Editor's Note. As pointed out by several solvers, the statement above must be corrected by adding the hypothesis that at least one of c, d be nonzero for every quadruple $(a, b, c, d) \in S$. We apologize for this omission.

Solution by Joel Schlosberg, Bayside, NY.
We prove the assertion under the weaker assumption that n is not a perfect square. Let M be any upper bound for both of the sets $\{|ad - bc| : (a, b, c, d) \in S\}$ and $\{|ac - nbd| : (a, b, c, d) \in S\}$. Suppose $(a, b, c, d) \in S$. We have $c^2 - nd^2 \neq 0$ inasmuch as c, d are not both zero and $\sqrt{n}$ is irrational (since n is not a perfect square), hence $|c^2 - nd^2| \geq 1$ since c, d, n are integers. From Brahmagupta's identity,

$$(a^2 - nb^2)(c^2 - nd^2) = (ac - nbd)^2 - n(ad - bc)^2$$

and the triangle inequality, we have

$$\begin{aligned} |a^2 - nb^2| &\leq |a^2 - nb^2| \cdot |c^2 - nd^2| \\ &= |(ac - nbd)^2 - n(ad - bc)^2| \\ &\leq |ac - nbd|^2 + n|ad - bc|^2 \leq M^2 + nM^2. \end{aligned}$$

Thus, $\{a^2 - nb^2 : (a, b, c, d) \in S\}$ is bounded.

Also solved by Jinwoo Hwang (Korea), Daniel López-Aguayo (Mexico), Nicholas C. Singer, and the proposer. There was one incomplete or incorrect solution.

Alternating between exponentials and trigonometrics **April 2015**

1967. *Proposed by Marcel Chirita, Bucharest, Romania.*

Let n be a positive integer. Determine all functions $f, g : \mathbb{R} \to \mathbb{R}$ with continuous derivatives f' and g' that satisfy the following conditions: For every real number x,

$$(f(x))^2 + (g(x))^2 = (f'(x))^2 + (g'(x))^2 \quad \text{and} \quad f(x) + g(x) = g'(x) - f'(x).$$

Moreover, the equation $f(x) = g(x)$ has $n + 1$ real roots with the smallest one being $x = 0$.

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI.
Let (Q) be the quadratic (first) equation and (L) the linear (second) equation above, which are satisfied by the functions f, g by hypothesis. Multiplying (Q) by 2, squaring both sides of (L) and subtracting gives

$$\begin{aligned} &(f(x) - g(x))^2 = (f'(x) + g'(x))^2 \\ &\qquad \Rightarrow \quad (f(x) - g(x) + f'(x) + g'(x))(f(x) - g(x) - f'(x) - g'(x)) = 0. \end{aligned}$$

Hence, for all $x \in \mathbb{R}$,

$$f(x) - g(x) = g'(x) + f'(x), \quad \text{or} \tag{1}$$

$$g(x) - f(x) = g'(x) + f'(x). \tag{2}$$

Note that either (1) or (2), together with (L), implies (Q). Let $0 = x_0 < x_1 < \cdots < x_n$ be the $n + 1$ solutions to $f(x) = g(x)$. Let $I_0 = (-\infty, 0]$, $I_{n+1} = [x_n, +\infty)$, and $I_j = [x_{j-1}, x_j]$ for $j = 1, 2, \ldots, n$. For x in the interior of any interval I_j $(0 \leq j \leq n + 1)$, we have $f(x) \neq g(x)$; hence, equations (1) and (2) cannot both hold. By continuity of f, g, f', g', it follows that in each interval I_j, either (1) holds throughout I_j, or (2) does.

The system {(L), (1)} has general solution:

$$f(x) = A\cos x + B\sin x,$$
$$g(x) = -B\cos x + A\sin x. \tag{3}$$

The system {(L), (2)} has general solution:

$$f(x) = Ce^{-x},$$
$$g(x) = De^{x}. \tag{4}$$

Therefore, in each of the intervals I_j, the functions f, g are of one of the forms (3) or (4) above. At each endpoint x_j $(0 \le j \le n)$, we have $f(x_j) = g(x_j)$, so either equation (1) or (2) plus (L) gives $f(x_j) = g'(x_j) = -f'(x_j)$. It follows that piecewise-defined functions f, g on $\mathbb{R}$ whose restriction to each interval I_j is of one of the forms (3), (4) necessarily have continuous first derivatives provided f, g are continuous at the points $x_0, x_1, \ldots, x_n$.

For f, g of the form (4), the equation $f(x) = g(x)$ has at most one real solution (except in the trivial case $C = D = 0$ when $f(x) = g(x)$ holds identically, contradicting its assumed finite number of solutions). Similarly, for f, g of the form (3), any infinite interval $(-\infty, t]$ or $[t, +\infty)$ contains infinitely many solutions to $f(x) = g(x)$. It follows that f, g must be of the form (4) in each of the intervals I_0, I_{n+1} and of the form (3) in each of the intervals $I_1, I_2, \ldots, I_n$. In fact, by uniqueness of solutions to the system {(L), (1)} on any interval I given the values $f(x)$, $g(x)$ at an arbitrary point $x \in I$, the functions f, g must be of the form (3) on the entire interval $[x_0, x_n] = I_1 \cup I_2 \cup \cdots \cup I_n$, for some A, B.

Since $x_0 = 0$, we must have $C = Ce^0 = f(0) = g(0) = De^{-0} = D$ in (4), so $f(x) = Ce^{-x}$ and $g(x) = Ce^x$ for some $C \neq 0$ and all $x \le 0$. Thus, we must have $A = f(0) = g(0) = -B$ in (3), so $f(x) = C(\cos x - \sin x)$ and $g(x) = C(\cos x + \sin x)$ for $0 \le x \le x_n$. Now, x_n is the n-th positive solution to $\cos x - \sin x = \cos x + \sin x$, i. e., $x_n = n\pi$ is the n-th positive root of the sine function. Clearly, $g(n\pi) = f(n\pi) = (-1)^n f(0) = (-1)^n C$. From (4), we see that $f(x) = (-1)^n Ce^{n\pi - x}$ and $g(x) = (-1)^n Ce^{x - n\pi}$ for $x \in I_{n+1} = [n\pi, +\infty)$. Summing up, the functions f, g are given by

$$f(x) = \begin{cases} Ce^{-x} & (x \le 0) \\ C(\cos x - \sin x) & (0 \le x \le n\pi) \\ (-1)^n Ce^{n\pi - x} & (x \ge n\pi), \end{cases} \quad g(x) = \begin{cases} Ce^{x} & (x \le 0) \\ C(\cos x + \sin x) & (0 \le x \le n\pi) \\ (-1)^n Ce^{x - n\pi} & (x \ge n\pi), \end{cases}$$

for an arbitrary constant $C \neq 0$.

Also solved by Dmitry Fleishman, Eugene A. Herman, and the proposer. There were 3 incomplete or incorrect solutions.

A trigonometric inequality for acute triangles **April 2015**

1968. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let n be a positive integer. Prove that in any triangle ABC,

$$\frac{\cos^n(\frac{A}{2})}{\sin^n(\frac{A}{2}) + \cos^n(\frac{A}{2})} + \frac{\cos^n(\frac{B}{2})}{\sin^n(\frac{B}{2}) + \cos^n(\frac{B}{2})} + \frac{\cos^n(\frac{C}{2})}{\sin^n(\frac{C}{2}) + \cos^n(\frac{C}{2})} \le \frac{3s^{n/3}}{r^{n/3} + s^{n/3}},$$

where s and r denote the semiperimeter and the inradius of $\triangle ABC$, respectively.

Editor's Note. Due to an editorial oversight, the statement of Problem 1968 failed to include the hypothesis that $\triangle ABC$ be acute. The solution presented below and other correct ones we received all assume this additional hypothesis. We apologize for the mistake.

Solution by Michel Bataille, Rouen, France.
Let $\alpha = A/2$, $\beta = B/2$, $\gamma = C/2$. Denote by L, R the left-hand and right-hand sides of the stated inequality. Clearly,

$$L = \frac{1}{1+\tan^n \alpha} + \frac{1}{1+\tan^n \beta} + \frac{1}{1+\tan^n \gamma}.$$

If $BC = a$, $CA = b$, $AB = c$, and $s = (a+b+c)/2$, then $\tan\alpha = r/(s-a)$, $\tan\beta = r/(s-b)$, $\tan\gamma = r/(s-c)$, and

$$\tan\alpha\tan\beta\tan\gamma = \frac{r^3}{(s-a)(s-b)(s-c)} = \frac{r}{s}$$

since the area of ABC is $[ABC] = rs = \sqrt{s(s-a)(s-b)(s-c)}$. It follows that

$$R = \frac{3}{1+\left(\tan\alpha\tan\beta\tan\gamma\right)^{n/3}}.$$

Consider the function $f(x) = 1/(1+e^{nx})$. We have

$$f''(x) = -n^2 e^{nx}(1-e^{nx})(1+e^{nx})^{-3} \le 0 \quad \text{when } x \le 0,$$

so f is concave on $(-\infty, 0]$. Since A, B, C are acute, the real numbers $\log\tan\alpha$, $\log\tan\beta$, $\log\tan\gamma$ are nonpositive. By Jensen's inequality,

$$\begin{aligned} L &= f(\log\tan\alpha) + f(\log\tan\beta) + f(\log\tan\gamma) \\ &\le 3f\left(\frac{\log\tan\alpha + \log\tan\beta + \log\tan\gamma}{3}\right) = R. \end{aligned}$$

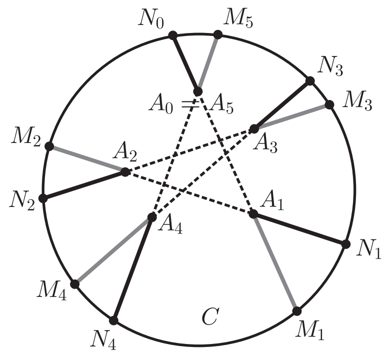
Also solved by Neculai Stanciu (Romania), Michael Vowe (Switzerland), and the proposer. There were two incomplete or incorrect solutions.

An equilateral polygon inside a circle **April 2015**

1969. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Let $n \ge 3$ be an integer, C be a circle, and $A_0, \ldots, A_{n-1}, A_n$ (with $A_n = A_0$), be an equilateral polygon (not necessarily convex or nonintersecting) inside C; that is, $A_0A_1 = A_1A_2 = \cdots = A_{n-1}A_n$. For $0 \le k < n$, the line A_kA_{k+1} intersects the circle C at two points; the one that belongs to the ray A_kA_{k+1} is denoted M_{k+1} and the other one is denoted by N_k. Prove that

$$\sum_{k=0}^{n-1} A_kN_k = \sum_{k=1}^{n} A_kM_k.$$



Solution by Northwestern University Math Problem Solving Group, Evanston, IL.
Let $A_{n+1} = A_1$, $M_n = M_0$, $M_{n+1} = M_1$, $N_n = N_0$, and $N_{n+1} = N_1$. For $k = 1, 2, \dots, n$, the segment $A_k A_{k+1}$ has fixed length l (independent of k); hence,

$$N_{k-1}A_k = N_{k-1}A_{k-1} + l \quad \text{and} \quad A_k M_{k+1} = A_{k+1}M_{k+1} + l.$$

By the intersecting chords theorem, we have

$$(N_{k-1}A_{k-1} + l) \cdot A_k M_k = N_{k-1}A_k \cdot A_k M_k = N_k A_k \cdot A_k M_{k+1} = N_k A_k \cdot (A_{k+1}M_{k+1} + l),$$

hence,

$$(N_k A_k - A_k M_k) \cdot l = N_{k-1}A_{k-1} \cdot A_k M_k - N_k A_k \cdot A_{k+1}M_{k+1}.$$

Adding the equations obtained by letting $k = 1, 2, \dots, n$, we have

$$\begin{aligned} l \cdot \sum_{k=1}^{n} (N_k A_k - A_k M_k) &= \sum_{k=1}^{n} (N_{k-1}A_{k-1} \cdot A_k M_k - N_k A_k \cdot A_{k+1}M_{k+1}) \\ &= N_0 A_0 \cdot A_1 M_1 - N_n A_n \cdot A_{n+1}M_{n+1} = 0. \end{aligned}$$

Since $l \neq 0$, we have $\sum_{k=1}^{n} (N_k A_k - A_k M_k) = 0$, so

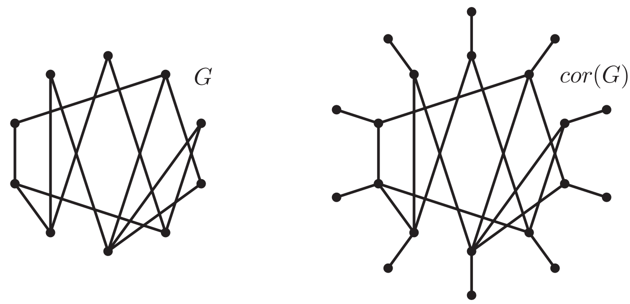
$$\sum_{k=0}^{n-1} N_k A_k = \sum_{k=1}^{n} N_k A_k = \sum_{k=1}^{n} A_k M_k.$$

Also solved by Michel Bataille (France), Robert L. Doucette, Marty Getz & Dixon Jones, Ahmad Habil (Syria), Eugene A. Herman, GWstat Problem Solving Group, Peter McPolin (Ireland), Jerry Minkus, Missouri State University Problem Solving Group, Joel Schlosberg and the proposer. There was 1 incomplete or incorrect solution.

The shifting aura of a graph's corona **April 2015**

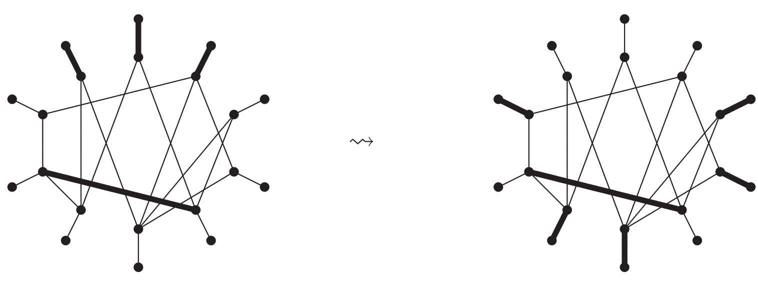
1970. *Proposed by Allen J. Schwenk, Western Michigan University, Kalamazoo, MI.*

A graph G has vertices $V(G) = \{v_1, v_2, \dots, v_n\}$ and edges $E(G) = \{e_1, e_2, \dots, e_m\}$. The *corona* of G, denoted by $cor(G)$, is the graph formed by adding n new vertices $w_1, w_2, \dots, w_n$ and n new edges $v_i w_i$ for $1 \leq i \leq n$. A set of edges is called *independent* if no two edges share a vertex. Let b_i denote the number of independent edge-sets of size i. Prove that for any graph G and for each i with $0 \le i < n/2$, the independent edge-sets of $cor(G)$ satisfy $b_i = b_{n-i}$.



Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

Call a vertex or an edge of $cor(G)$ *original* if it belongs to the original graph G and *coronal* otherwise. Let S be an independent edge-set of size i. If S contains j original edges, then the other $i - j$ edges in S are coronal. Thus, the edges of S are incident to $2j + (i - j) = i + j$ original vertices (which are all distinct, by independence). Taking the j original edges in S together with the $n - (i + j)$ coronal edges that are each incident to (exactly) one of the $n - (i + j)$ original vertices not incident to any edge in S, we obtain an independent edge-set S' of size $j + [n - (i + j)] = n - i$. The figure below shows an example of the passage from S (whose edges are the solid lines on the left graph) to S' (whose edges are the solid lines on the right graph).



The transformation $S \leftrightarrow S'$ is invertible; in fact, it is an involution since clearly $(S')' = S$. Thus, we have constructed a bijection between the collection of independent edge-sets of size i and the collection of independent edge-sets of size $n - i$. This proves that $b_i = b_{n-i}$.

Also solved by Joseph DiMuro, Christopher J. D. Dowd, Dmitry Fleischman, Natacha Fontez-Merz, Raymond N. Greenwell, Jerrold W. Grossman, Rob Pratt, Joel Schlosberg, John H. Smith and the proposer.

Answers

Solutions to the Quickies from page 224.

A1061. We have $x^3(1 - y^2) = y^3(x^2 - 1)$, so $x^3 + y^3 = x^2y^2(x + y)$. Since $x + y > 0$, we conclude that $x^2 - xy + y^2 = x^2y^2$; hence, $(1 - x^2)(1 - y^2) = 1 - xy$. It follows that

$$\frac{xy-1}{x^3y^3} = \frac{x^2-1}{x^3} \cdot \frac{1-y^2}{y^3} = z^2,$$

which implies $xy = 1 + z^2x^3y^3$.

A1062. For $x \in (0, \pi/2)$, let $F(x) = \int_{\tan x}^{\cot x} (1+t^2)^{-2} dt$. Let G be an antiderivative of $t \mapsto (1+t^2)^{-2}$. By the fundamental theorem of calculus, $F(x) = G(\cot x) - G(\tan x)$. Differentiating, we get

$$F'(x) = G'(\cot x)(-\csc^2 x) - G'(\tan x)\sec^2 x = -\frac{\csc^2 x}{(1+\cot^2 x)^2} - \frac{\sec^2 x}{(1+\tan^2 x)^2}$$
$$= -\frac{1}{\csc^2 x} - \frac{1}{\sec^2 x} = -(\sin^2 x + \cos^2 x) = -1.$$

Since $F(\pi/4) = 0$, we have $F(x) = \pi/4 - x$ for $x \in (0, \pi/2)$. Thus,

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dt}{(1+t^2)^2} = F\left(\frac{\pi}{6}\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

Ask Siri . . .

A: What is 10 divided by 0?
S: Imagine that you have 10 cookies and you split them evenly among 0 friends. How many cookies does each of your friends get? See, it doesn't make sense. So Cookie Monster gets them all. Nom nom nom.

$$10 \div 0 = \text{undefined}.$$

Submitted by Anneliese Jones, Ann Arbor, MI

Math Joke

Let dog $= \emptyset$.

Q: What's sup(dog)?

A: Nothing.

— Amanda Colon, undergraduate student, Gordon College
Communicated by Karl-Dieter Crisman, Gordon College

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Csicsery, George, *Navajo Math Circles*, Zala Films, 2016; video, 58:03, plus bonus features, 34:42. Colleges/libraries (includes performance rights), $149; home video/personal use, $24.95; plus $5 shipping/handling. Order form at http://www.zalafilms.com/navajo/nmcorder.pdf. ISBN 978-0-98248-006-9.

This video vividly brings to life the inspiring student and teacher math circles and a two-week summer math camp on the Navajo reservation. Math circles use no textbooks but share enthusiasm for mathematics through open-ended questions that can lead to deep mathematics; as one student notes, "The answers actually don't matter." Notable are the contributions of Tatiana Shubin (San Jose State University) and Henry H. Fowler (Diné College), who started these circles in 2012. The film interviews instructors, teachers, and students (and their parents) and shows classroom sessions and outside activities. The students are expressively overjoyed by the experience, and the parents are enthusiastic. Students are shown working on problems and confidently attacking them: the number of diagonals of an icosahedron, patterns in billiard-ball reflections, arranging touching pennies, tiling with dominoes, "boomerang" fractions. The circles and camp also include a component of Navajo language and culture, with "the math portion stemming from who we are as Navajo people." Though that connection is not obvious from the problems shown, a strong common thread in mathematics and Navajo culture is love of beauty (*hózhó*). "Math camp helped me to stay closer to home ... it was an enrichment for the mind ... you don't have to go somewhere [else] to be important." The bonus features include some of the life stories of Shubin and Fowler. (Thanks to Phil Straffin.)

Roberts, Gareth E., *From Music to Mathematics: Exploring the Connections*, Johns Hopkins University Press, 2016; xviii + 301 pp, $49.95. ISBN 978-1-4214-1918-3.

This book arose from the author's year-long introductory course in math and music. Both musical notation and geometric series make their appearance in the first chapter, so do least common multiples and recurrence relations. Chapter 2 is devoted to music theory (cyclic shifts occur), chapter 3 to sound (trigonometric functions and identities, the harmonic oscillator, and beats), and chapter 4 to tuning (continued fractions arise). Subsequent chapters (covered in the second semester) treat musical group theory, change ringing, 12-tone music, and "mathematical modern music." Each section ends with exercises (solutions not provided).

Suri, Manil, 90th-birthday party, *New York Times* (25 April 2016) A15. http://www.nytimes.com/2016/04/25/opinion/the-mathematicians-90th-birthday-party.html.

Is a mathematician "over the hill" by 40? "Strides have been made in addressing the sexism in the field but what about the ageism?" Author Suri suggests that, particularly in applied mathematics, "experience and maturity" are valuable, not to mention "the ability to interact with non-mathematicians, to interpret their questions mathematically and to explain solutions in their language." The occasion of this Op Ed piece is the 90th birthday of a mathematics colleague who at age 70 published a paper on a method (still used by engineers) to design machine parts.

Math. Mag. **89** (2016) 231–232. doi:10.4169/math.mag.89.3.231.

Chalkdust. Issues 01–03 (Spring 2015—Spring 2016). http://chalkdustmagazine.com.

Hurrah! There is now another worthwhile magazine about mathematics for undergraduates. *Chalkdust* is a twice-yearly new mathematics magazine put out by students in the Dept. of Mathematics at University College London. It features articles, interviews, and cartoons by faculty and students. The content is interesting to—and suitable for—undergraduates, and the content—plus an attractive layout with plenty of color photos and illustrations—makes *Chalkdust* fun to peruse and worthwhile to read. Printed copies are available in the U.K. only, but the entire magazine is available free at the Website.

Jeffries, Stuart, Genius by numbers: Why Hollywood maths movies don't add up, *The Guardian* (6 April 2016) http://www.theguardian.com/film/2016/apr/06/mathematics-movies-the-man-who-knew-infinity.

Author Stuart points out instances of Hollywood's penchant for having "math geniuses" write on glass, thus purporting to exhibit dramatically more passion than writing on paper. Despite other praise, he accuses the new film *The Man Who Knew Infinity* (about Ramanujan, released in April) of "leaving a black hole where the maths should be." Still, he understands the difficulty for films of "dramatising difficult ideas." So what's a filmmaker to do? Stuart cites humorous instances of film scenes that offer a "parody of explanation" that "mocks you ... and your intellectual aspirations." His last sentence puts the marker to the glass: "You are never going to understand how difficult stuff works from watching movies, however much you'd like to."

Weisberg, Herbert I., *Willful Ignorance: The Mismeasure of Uncertainty*, Wiley, 2014; xvii + 434 pp, $34.95 (P). ISBN 978-0-470-89044-8.

"[W]illful ignorance is the central concept that underlies mathematical probability." Author Weisberg refers to stripping away particularities on the way to a mathematical model. But he asks, Why has scientific research "contributed so little to our understanding of 'what works' in areas like education, health care, and economic development"? He goes on to lament that the emphasis today on statistical methodology "displays a fundamental *lack* of trust in the validity of insight and intuition." He suggests greater ambiguity in research but a confirmatory role for probability-based methods, with the slogan "Trust, but verify." The book is both a history of probability and an argument to change practices in the medical and social sciences.

Farris, Frank A., *Creating Symmetry: The Artful Mathematics of Wallpaper Patterns*, Princeton University Press, 2015; xiii + 230 pp, $35. ISBN 978-0-691-16173-0.

You may know that there are only seven frieze patterns and only 17 wallpaper patterns in the plane and may even have taught students to identify their symmetries. But you have never seen those entities treated in the fashion of this book. Despite its disclaimer that "the mathematical prerequisite is some knowledge of calculus," the book is a treasure trove of all kinds of mathematics brought to bear on symmetry: vector spaces, Fourier series, complex analysis, vector calculus, eigenfunctions of the Laplacian, "wallpaper waves," quadratic number fields, color-turning and color-reversing wallpaper functions, and hyperbolic wallpaper (just in case you live in a hyperbolic house). This is a rich feast fit for a senior mathematics major; there are even exercises. Moreover, the color illustrations are stupendous. Beautiful mathematics! Beautiful book! (My only wish is that the lovely wallpaper of the dust cover had found its way to an antique marbling of the book cover itself.)

Ornes, Stephen, The whole universe catalog, *Scientific American* (July 2015) 68–75.

Mathematicians behind the 1980 proof of the complete classification of finite simple groups are getting old. Does anyone else understand the proof, which extends through 500 articles and 15,000 journal pages? Once those individuals are gone and no one else understands the proof, can we still regard the theorem as proved? Four of them put together an outline (only 350 pages!), *The Classification of Finite Simple Groups* (AMS, 2011), in an effort to pass the proof on. This is a story about the people; there's not much mathematics in this article (*Scientific American* forbids equations), just a sidebar describing briefly the four families of groups involved. Will there be a "second-generation" proof that also generates new far-reaching ideas?

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